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By

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Directional convergence of spectral regularization method associated to families of closed operators*

Gisela L. Mazzieri[†‡] Ruben D. Spies[✉ †§] Karina G. Temperini[†¶]

Abstract

We consider regularized solutions of linear inverse ill-posed problems obtained with generalized Tikhonov-Phillips functionals with penalizers given by linear combinations of seminorms induced by closed operators. Convergence of the regularized solutions is proved when the vector regularization rule approaches the origin through appropriate radial and differentiable paths. Characterizations of the limiting solutions are given. Finally, a examples of image restoration using generalized Tikhonov-Phillips methods with convex combinations of seminorms are shown.

Keywords: Inverse problem, Ill-Posed, Regularization, Tikhonov-Phillips, closed operators.

AMS Subject classifications: 47A52, 65J20

1 Introduction

Very often an inverse problem can be formulated as the necessity of approximating x in an equation of the form

$$Tx = y, \tag{1}$$

where T is a linear bounded operator between two infinite dimensional Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$ (in general these will be function spaces), the range of T, $\mathcal{R}(T)$, is non-closed and y is the data, supposed to be known, perhaps with a certain degree of error. It is well known that under these hypotheses, problem (1) is ill-posed in the sense of Hadamard ([4]). In this case the ill-posedness is a result of the unboundedness of $T^\dagger$, the Moore-Penrose inverse of T. The Moore-Penrose inverse is a fundamental tool in the treatment of inverse ill-posed problems and their regularized solutions. This is so mainly because the least-squares solution

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of minimum norm of problem (1), also known as the best approximate solution, is precisely given by $x^\dagger \doteq T^\dagger y$, which exists if and only if $y \in \mathcal{D}(T^\dagger) = \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$. Moreover, for any given $y \in \mathcal{D}(T^\dagger)$, the set of all least-squares solutions of problem (1) is given by $x^\dagger + \mathcal{N}(T)$, where $\mathcal{N}(T)$ denotes the null space of the operator T.

Since $T^\dagger$ is unbounded, small errors or noise in the data y may induce arbitrarily large errors in the corresponding approximated solutions (see [13], [12]), thus turning unstable all standard numerical approximation methods, making them unsuitable for most applications and inappropriate from any practical point of view. The so called "regularization methods" are mathematical tools designed to restore stability to the inversion process and consist essentially of parametric families of continuous linear operators approximating $T^\dagger$. The mathematical theory of regularization methods is very wide (a comprehensive treatise on the subject can be found in the book by Engl, Hanke and Neubauer, [3]) and it is of great interest in a broad variety of applications in many areas such as Medicine, Physics, Geology, Geophysics, Biology, image restoration and processing, etc.

There are many ways of regularizing an ill-posed inverse problem. Among the most standard and traditional ones we mention the Tikhonov-Phillips method ([11], [14], [15]), truncated singular value decomposition (TSVD), Showalter's method, total variation regularization ([1]), etc. However, the best known and most commonly and widely used is without a doubt the Tikhonov-Phillips regularization method, which was originally and independently proposed by Tikhonov and Phillips in 1962 and 1963 (see [11], [14], [15]). Although this method can be formalized within a very general framework by means of spectral theory ([3], [2]), the widespread of its use is undoubtedly due to the fact that it can also be formulated in a very simple way as an optimization problem. In fact, the regularized solution of problem (1) obtained by applying the classical Tikhonov-Phillips method is the minimizer x_α of the functional

$$J_\alpha(x) \doteq \|Tx - y\|^2 + \alpha \|x\|^2, \tag{2}$$

where α is a positive constant known as the regularization parameter.

The penalizing term $\alpha \|x\|^2$ in (2) not only induces stability but it also determines certain regularity properties of the approximating regularized solutions x_α and of the corresponding least-squares solution which they approximate as the regularization parameter α approaches 0^+. Thus, for instance, it is well known that minimizers of (2) are always "smooth" and, for $\alpha \to 0^+$, they approximate the least-squares solution of minimum norm of (1), that is $\lim_{\alpha\to 0^+} x_\alpha = T^\dagger y$. This method is known as the Tikhonov-Phillips method of order zero. Choosing other penalizers gives rise to different approximations with different properties, approximating different least-squares solutions of (1). Thus, for instance, the use of $\|\nabla x\|^2$ as penalizer instead of $\|x\|^2$ in (2) gives rise to the so called Tikhonov-Phillips method of order one, the penalizer $\|x\|_{\mathrm{BV}}$ (where $\|\cdot\|_{\mathrm{BV}}$ denotes the bounded variation norm) originates the so called bounded variation regularization method introduced by Acar and Vogel in 1994 ([1]), etc. In particular, in the latter case, the approximating solutions are only forced to be of bounded variation rather than smooth and they approximate, for $\alpha \to 0^+$, the least-squares solution of problem (1) of minimum $\|\cdot\|_{\mathrm{BV}}$-norm (see [1]). This method has been proved to be a good choice in certain image restoration problems in which it is highly desirable to preserve sharp edges and discontinuities of the original image.

Thus, the penalizing term in (2) is used not only to stabilize the inversion of the ill-posed problem but also to enforce certain characteristics of the approximating solutions and of the particular limiting least-squares solution that they approximate. Hence, it is reasonable to assume that an adequate choice of the penalizer, based on *a-priori* knowledge about certain characteristics of the exact solution of problem (1), will lead to approximated "regularized"

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solutions which will appropriately reflect those characteristics.

For the case of Tikhonov-Phillips functionals with a general penalizer W, i.e.

$$J_{W,\alpha}(x) \doteq \|Tx - y\|^2 + \alpha W(x) \quad x \in \mathcal{D}, \tag{3}$$

where $W(\cdot)$ is an arbitrary functional with domain $\mathcal{D} \subset \mathcal{X}$ and α is a positive constant, sufficient conditions on W guaranteeing existence, uniqueness and stability of the minimizers where found in [10].

In this article we study the case in which the penalizer W in (3) is given by $W(x) \doteq \sum_{i=1}^{N} \alpha_i \|L_i x\|^2$, where $\alpha_i > 0\ \forall i = 1, 2, \ldots, N$, and the L_i's are operators satisfying certain hypotheses. For these cases we analyze the convergence of the minimizers as the vector regularization rule $\vec{\alpha} \doteq (\alpha_1, \alpha_2, \ldots, \alpha_N)^T$ approaches $\vec{0}$ through appropriate paths. We will also characterize the limiting least-squares solutions. Finally, several examples consisting of applications to image restoration are presented.

2 Preliminaries

The so called "best approximate solution" $x^\dagger$ of problem (1) is defined as the least-squares solution on minimum norm. Thus, $x^\dagger$ satisfies:

(i) $\left\|Tx^\dagger - y\right\| = \inf\{\|Tz - y\| : z \in \mathcal{X}\}$,

(ii) $\left\|x^\dagger\right\| = \inf\{\|z\| : z$ is a least-squares solution of $Tx = y\}$.

It is a well known fact that $x^\dagger$ exists if and only if $y \in \mathcal{D}(T^\dagger) = \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$, in which case it is given by $x^\dagger = T^\dagger y$. When T is not injective, choosing the minimum norm solution is a way of forcing uniqueness of solutions. In some cases, however, this may not be the best choice. For instance, one could be interested in selecting the least-squares solution that minimizes the seminorm induced by a certain operator L, i.e., find $x_L^\dagger$, least-squares solution of (1) such that

$$\left\|Lx_L^\dagger\right\| = \inf\{\|Lz\| : z \text{ is a least-squares solution of } Tx = y\},$$

where L is a given operator on a certain domain $\mathcal{D} \subset \mathcal{X}$. From a purely mathematical point of view, the characterization of such a least-squares solution can be done via the weighted generalized inverse of T (see [3]). Independently of the operator L, however, approximating $x_L^\dagger$ is still an unstable problem, requiring regularization. With that in mind we propose the following minimization problem:

$$\min_{x \in \mathcal{D}(L)} \|Tx - y\|^2 + \alpha \|Lx\|^2 . \tag{4}$$

Clearly, a solution of (4), if it exists, belongs to $\mathcal{D}(L)$. Hence, the use of $\|Lx\|^2$ as a penalizer is only appropriate under such "*a-priori*" knowledge about the exact solution. When that assumption is uncertain one can still use $\|Lx\|^2$ as a penalizer by considering the Hilbert scale induced by L over $\mathcal{X}$ (see [3] and also [9]).

Throughout this section we will suppose that L is a linear closed, densely defined operator mapping $\mathcal{D}(L) \subset \mathcal{X}$ *onto* a Hilbert space $\mathcal{Z}$ (often L is a differential operator) satisfying the following "complementation condition":

$$\text{(CC)} \qquad \exists\, \gamma > 0 \text{ such that } \|Tx\|^2 + \|Lx\|^2 \geq \gamma \|x\|^2 \quad \forall\, x \in \mathcal{D}(L).$$

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Note that condition (CC) implies $\mathcal{N}(T)\cap\mathcal{N}(L)=\{0\}$. It is easy to prove that if $dim\,\mathcal{N}(L)<\infty$, then the condition $\mathcal{N}(L)\cap\mathcal{N}(T)=\{0\}$ is also sufficient for (CC). This is particularly important when L is a differential operator.

We now define a new inner product and a "weighted" norm on $\mathcal{D}(L)$ by:

$$\langle x,\widehat{x}\rangle_{TL}\doteq\langle Tx,T\widehat{x}\rangle+\langle Lx,L\widehat{x}\rangle,\qquad \|x\|_{TL}\doteq\langle x,x\rangle_{TL}^{1/2},\quad x\in\mathcal{D}(L). \tag{5}$$

It can be easily proved that $\mathcal{D}(L)$, equipped with this TL-inner product is a Hilbert space (see [3]) that we shall denote by $\mathcal{X}_{TL}$. Throughout the rest of this section the subscript "TL" will always make reference to this space.

Consider now the operator T_L defined as the restriction of T to $\mathcal{D}(L)$, that is,

$$\begin{aligned} T_L:\mathcal{X}_{TL}\doteq(\mathcal{D}(L),\langle\cdot,\cdot\rangle_{TL}) &\longrightarrow \mathcal{Y}\\ x &\longrightarrow Tx \end{aligned} \tag{6}$$

We shall denote with $L^\dagger_{TL}$ and $T^\dagger_{TL}$ the Moore-Penrose inverses of $L:\mathcal{X}_{TL}\longrightarrow\mathcal{Z}$ and $T_L:\mathcal{X}_{TL}\longrightarrow\mathcal{Y}$, respectively. It is timely to point out that $T^\dagger_{TL}$ and $L^\dagger_{TL}$ are, in general, different from the generalized inverses $T^\dagger$ and $L^\dagger$, respectively. We shall refer to the former ones as the "weighted generalized inverses", to distinguish them from the latter ones and to emphasize the fact that they are obtained by considering the inner product "weighted" by the operators T and L, defined in (5).

We will also need to consider the operator $T_0\doteq T_{|\mathcal{N}(L)}$. This operator will play an important role in the definition of a regularization family of operators that we will introduce later on, since $T_0^\dagger$, the Moore-Penrose inverse of T_0 is bounded. Note also that the generalized inverses $T_0^\dagger$ and $T^\dagger_{0,TL}$ are equal since T_0 is injective.

The following fundamental result relates the least-squares solutions of (1) with the weighted generalized inverse $T^\dagger_{TL}$.

Theorem 2.1. *Let $y\in\mathcal{D}(T^\dagger_{TL})$. Then $x^\dagger_L\doteq T^\dagger_{TL}y$ is a least-squares solution of $T_Lx=y$ and for any other least-squares solution $\tilde{x}$ there holds*

$$\left\|Lx^\dagger_L\right\|<\|L\tilde{x}\|.$$

Also, if the range of T is not closed then the operator $T^\dagger_{TL}$ is unbounded.

Proof. See [3]. □

Remark 2.2. *Note here that if $\mathcal{N}(T)\cap\mathcal{N}(L)$ was not trivial then the solution $x^\dagger_L$ characterized by the previous theorem would not be unique. It is also important to note that no selection of L can transform problem (1) into a well-posed problem. In fact the ill-posedness is a consequence of the fact that the range of T is not closed.*

Having defined and characterized the operator $T^\dagger_{TL}$ we are now interested in finding appropriate regularizations. For this purpose we could, in principle use all classical regularization methods considering the operator T defined on the Hilbert space $\mathcal{X}_{TL}$ and define a family of regularization operators R_α as $R_\alpha\doteq g_\alpha(T^\sharp T)T^\sharp$, given an appropriately chosen family of functions g_α, where $T^\sharp$ denotes the adjoint of T_L in the TL-topology. This approach, for the traditional Tikhonov-Phillips method, was studied by Locker and Prenter in [8]. From the computational point of view, the approach presents some disadvantages since it requires the computation of the adjoint operator $T^\sharp=(T^*T+L^*L)^{-1}T^*$ (see [8]). However, there exists a way of regularizing $T^\dagger_{TL}$ without having to compute the adjoint operator $T^\sharp$, as the next theorem shows.

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Theorem 2.3. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ Hilbert spaces, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $T^\dagger$ the Moore-Penrose generalized inverse of T, $L : \mathcal{D}(L) \subset \mathcal{X} \longrightarrow \mathcal{Z}$ a linear, densely defined, close operator, $L^\dagger_{TL}$ the Moore-Penrose generalized inverse of L on $\mathcal{X}_{TL}$, T_L as in (6) y $B \doteq T_L L^\dagger_{TL}$. Let $g_\alpha : [0, \|B\|^2] \to \mathbb{R}$, $\alpha > 0$, be a family of functions satisfying the following conditions:*
(C1) For every $\alpha \in (0, \alpha_0)$, $g_\alpha(\lambda)$ is piecewise continuous for $\lambda \in [0, +\infty)$ and continuous from the right at points of discontinuity.
(C2) There exists a constant $C > 0$ (independent of α) such that $|\lambda g_\alpha(\lambda)| \le C$ for every $\lambda \in [0, +\infty)$, for every $\alpha \in (0, \alpha_0)$.
(C3) For every $\lambda \in (0, +\infty)$, $\lim_{\alpha \to 0^+} g_\alpha(\lambda) = \frac{1}{\lambda}$.

For $y \in \mathcal{D}(T^\dagger_{TL})$ we define the regularized solution of problem (1) by

$$R_\alpha y \doteq T_0^\dagger + L^\dagger_{TL} g_\alpha(B^* B) B^* y. \tag{7}$$

Then for every $y \in \mathcal{D}(T^\dagger_{TL})$ there holds

$$R_\alpha y \to T^\dagger_{TL} y, \quad L R_\alpha y \to L T^\dagger_{TL} y, \quad T R_\alpha y \to Q y,$$

as $\alpha \to 0^+$ (here Q is the orthogonal projection of $\mathcal{Y}$ onto $\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(T_L)}$). If $y \notin \mathcal{D}(T^\dagger_{TL})$, then $\lim_{\alpha \to 0^+} \|L R_\alpha y\| = \infty$.

Proof. See [3]. □

Note that the convergence result of Theorem 2.3 is equivalent to convergence in the norm of the graph of the operator L, defined on $\mathcal{D}(L)$ as $\|x\|_L^2 \doteq \|x\|^2 + \|Lx\|^2$, which is clearly stronger than the original norm in $\mathcal{X}$.

In the following proposition a relation between the regularized solutions defined in (7) and a generalized Tikhonov-Phillip method with penalizer $\|Lx\|^2$ is shown.

Proposition 2.4. *Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, T, $T^\dagger$, L, $L^\dagger_{TL}$, T_L, $B = T_L L^\dagger_{TL}$ and R_α, all as in Theorem 2.3. Also for $y \in \mathcal{D}(T^\dagger_{TL})$, let $x_\alpha \doteq R_\alpha y = T_0^\dagger y + L^\dagger_{TL} g_\alpha(B^* B) B^* y$ with $g_\alpha(\lambda) \doteq \frac{1}{\lambda + \alpha}$. Then for each fixed $\alpha > 0$, x_α is the unique global minimizer of the generalized Tikhonov-Phillips functional*

$$\begin{aligned} J_\alpha : \mathcal{D}(L) &\longrightarrow \mathbb{R}^+ \\ x &\longrightarrow \|T_L x - y\|^2 + \alpha \|Lx\|^2. \end{aligned} \tag{8}$$

Proof. See [3]. □

Remark 2.5. *Since the family of functions $g_\alpha(\lambda) = \frac{1}{\lambda + \alpha}$ clearly satisfies the hypotheses of Theorem 2.3, it then follows from Proposition 2.4 that the regularized solutions obtained with the generalized Tikhonov-Phillip method with penalizer $\|Lx\|^2$ converge to $T^\dagger_{TL} y$ as $\alpha \to 0^+$ provided that $y \in \mathcal{D}(T^\dagger_{TL})$.*

In light of the previous analysis and results one sees that the penalizing term in (8), on one hand induces stability and on the other hand it allows the approximation of $x_L^\dagger$ in such a way that the approximated regularized solutions share with the exact solution certain properties or characteristics that one presumes that such a solution possesses. Hence, it is reasonable to assume that an adequate choice of the penalizer, based on the *"a-priori"* knowledge of certain type of information about the exact solution, will result in approximated solutions which appropriately reflect those characteristics. Following this line of reasoning it is also reasonable to assume that the simultaneous use of two or more penalizers of different nature

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will, in some way, allow the capturing of different characteristics on the exact solution. This is particularly relevant, for instance, in image restoration problems in which it is known *"a-priori"* that the original image is "blocky", i.e. it possesses both regions of high regularity and regions with sharp discontinuities. In the following section we shall extend the results of Theorem 2.3 and Proposition 2.4 to this type of penalizers. It is important to note however that the regularization parameter will now be vector-valued.

3 Penalization with linear combination of semi-norms associated to closed operators

We study here the case of generalized Tikhonov-Phillips regularization methods for which the penalizing terms in (8) is of the form $W(x) \doteq \sum_{i=1}^{N} \alpha_i \|L_i x\|^2$, where the L_i's are closed linear operators, i.e. we consider functionals of the form

$$J_{\vec{\alpha},L_1,L_2,\dots,L_N}(x) \doteq \|Tx - y\|^2 + \sum_{i=1}^{N} \alpha_i \|L_i x\|^2. \tag{9}$$

The following results (which can be found in [10]) establish conditions guaranteeing existence, uniqueness and strong stability of the global minimizers of the functional (9).

Theorem 3.1. *Let $\mathcal{X}$, $\mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_N$ be reflexive Banach spaces, $\mathcal{Y}$ a normed space, $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, $\mathcal{D}$ a subspace of $\mathcal{X}$, $L_i : \mathcal{D} \longrightarrow \mathcal{Z}_i$, $i = 1, 2, ..., N$, closed linear operators with $\mathcal{R}(L_i)$ weakly closed for every $1 \le i \le N$ and such that $T, L_1, L_2, \dots, L_N$ are complemented, i.e. there exists a constant $k > 0$ such that $\|Tx\|^2 + \sum_{i=1}^{N} \|L_i x\|^2 \ge k\|x\|^2$, $\forall x \in \mathcal{D}$. Then, for any $y \in \mathcal{Y}$, $\alpha_1, \alpha_2, \dots, \alpha_N \in \mathbb{R}^+$ the functional $J_{\vec{\alpha},L_1,L_2,\dots,L_N}(\cdot)$ given in (9) has a unique global minimizer.*

Proof. See [10]. □

Under the same hypotheses of Theorem 3.1 one has that the minimizer of (9) is stable under perturbations in the data y, in the parameters α_i and in the model operator T. Before we proceed to the statements of this results, we shall need the following definition.

Definition 3.2. *(W-uniform consistency) Let $\mathcal{X}$, $\mathcal{Y}$ be vector spaces, $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ and $W, F, F_n, n = 1, 2, ...,$ functionals defined on a set $\mathcal{D} \subset \mathcal{X}$. We will say that the sequence $\{F_n\}$ is W-uniformly consistent for F if $F_n \to F$ uniformly on every W-bounded set, that is if for any given $c > 0$ and $\epsilon > 0$, there exists $N = N(c, \epsilon)$ such that $|F_n(x) - F(x)| < \epsilon$ for every $n \ge N$ and every $x \in \mathcal{D}$ such that $|W(x)| \le c$.*

Lemma 3.3. *Let all the hypotheses of Theorem 3.1 hold. Let also $\vec{L} \doteq (L_1, L_2, \dots, L_N)^T$, $y, y_n \in \mathcal{Y}$, $T_n \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, $n = 1, 2, \dots,$ such that $y_n \to y$, $\{T_n\}$ is $\vec{L}$-uniformly consistent for T and for each $i = 1, 2, \dots, N$, let $\{\alpha_i^n\}_{n=1}^{\infty} \subset \mathbb{R}^+$ such that $\alpha_i^n \to \alpha_i$ as $n \to \infty$. If x_n is a global minimizer of the functional*

$$J_n(x) \doteq \|T_n x - y_n\|^2 + \sum_{i=1}^{N} \alpha_i^n \|L_i x\|^2, \tag{10}$$

then $x_n \to \bar{x}$, where $\bar{x}$ is the unique global minimizer of (9).

Proof. See [10]. □

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3.1 Radial convergence of spectral methods

Let $\mathcal{X}, \mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_N$ Hilbert spaces, $\mathcal{D}$ a dense subspace of $\mathcal{X}$, $L_i : \mathcal{D} \longrightarrow \mathcal{Z}_i$, $i = 1, 2, ..., N$, linear, closed surjective operators such that the operator $L : \mathcal{X} \longrightarrow \bigotimes_{i=1}^{N} \mathcal{Z}_i$ defined by $L \doteq (L_1, L_2, \dots, L_N)^T$ has closed range. Suppose also that L satisfies the following complementation condition:

$$(CC): \; \exists\, \gamma > 0 \text{ such that } \|T_L x\|^2 + \|Lx\|^2 \geq \gamma \|x\|^2 \quad \forall\, x \in \mathcal{D},$$

or equivalently

$$\exists\, \gamma > 0 \text{ such that } \|T_L x\|^2 + \sum_{i=1}^{N} \|L_i x\|^2 \geq \gamma \|x\|^2 \quad \forall\, x \in \mathcal{D}, \tag{11}$$

where the operator T_L is defined as $T_L \doteq T_{|\mathcal{D}}$. Let also $\vec{\alpha} \doteq \alpha\vec{\eta}$, where $\alpha \in \mathbb{R}^+$ and $\vec{\eta} \doteq (\eta_1, \eta_2, \dots, \eta_N)^T \in \mathbb{R}^N_+$ such that $\sum_{i=1}^{N} \eta_i = 1$. Define the operator $L_{\vec{\eta}} : \mathcal{D} \longrightarrow \bigotimes_{i=1}^{N} \mathcal{Z}_i$, $L_{\vec{\eta}} \doteq \left(\sqrt{\eta_1} L_1, \sqrt{\eta_2} L_2, \dots, \sqrt{\eta_N} L_N\right)^T$ and a new weighted inner product and its associated norm on $\mathcal{D}$ as:

$$\langle x, \widehat{x} \rangle_{TL_{\vec{\eta}}} \doteq \langle T_L x, T_L \widehat{x} \rangle + \langle L_{\vec{\eta}} x, L_{\vec{\eta}} \widehat{x} \rangle, \qquad \|x\|_{TL_{\vec{\eta}}} \doteq \langle x, x \rangle^{1/2}_{TL_{\vec{\eta}}}, \quad x, \widehat{x} \in \mathcal{D}. \tag{12}$$

It can be easily proved that $\mathcal{X}_{TL_{\vec{\eta}}} \doteq (\mathcal{D}, \|\cdot\|_{TL_{\vec{\eta}}})$ is a Hilbert space. Denote with $L^\dagger_{TL_{\vec{\eta}}}$ the Moore-Penrose inverse of the operator $L_{\vec{\eta}}$ on $\mathcal{D}$ with this new $TL_{\vec{\eta}}$-inner product, i.e. consider $L_{\vec{\eta}}$ as an operator from $\mathcal{X}_{TL_{\vec{\eta}}}$ into $\mathcal{Y}$, and let $B_{\vec{\eta}}$ and T_0 the operators defined by $B_{\vec{\eta}} : \bigotimes \mathcal{Z}_i \longrightarrow \mathcal{Y}$, $B_{\vec{\eta}} \doteq TL^\dagger_{TL_{\vec{\eta}}}$ y $T_0 \doteq T_{|\mathcal{N}(L)}$.

The following theorem generalizes the result given by Theorem 2.3 to the case of a penalizer given by a linear combination of seminorms induced by closed operators.

Theorem 3.4. *Let $\{g_\alpha\}$ be a spectral regularization method, $\{R_{\alpha,\vec{\eta}}\}_{\alpha \in \left(0, \|B_{\vec{\eta}}\|^2\right)}$ a family of operators from $\mathcal{Y}$ into $\mathcal{X}$ defined by*

$$R_{\alpha,\vec{\eta}} \doteq T_0^\dagger + L^\dagger_{TL_{\vec{\eta}}} g_\alpha(B^*_{\vec{\eta}} B_{\vec{\eta}}) B^*_{\vec{\eta}}. \tag{13}$$

Then $\{R_{\alpha,\vec{\eta}}\}_{\alpha \in \left(0, \|B_{\vec{\eta}}\|^2\right)}$ is a family of regularization operators for $T^\dagger_{TL_{\vec{\eta}}}$. In particular for every $y \in \mathcal{D}(T^\dagger_{TL_{\vec{\eta}}})$ there holds $\lim_{\alpha \to 0^+} R_{\alpha,\vec{\eta}}\, y = T^\dagger_{TL_{\vec{\eta}}} y$, $\lim_{\alpha \to 0^+} L_{\vec{\eta}} R_{\alpha,\vec{\eta}}\, y = L_{\vec{\eta}} T^\dagger_{TL_{\vec{\eta}}} y$ and $\lim_{\alpha \to 0^+} T_L R_{\alpha,\vec{\eta}}\, y = Qy$, where Q is the orthogonal projection of $\mathcal{Y}$ onto $\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(T_L)}$.

Proof. Clearly the operator $L_{\vec{\eta}}$ is linear. We will prove that $L_{\vec{\eta}}$ satisfies the complementation condition. For this note that for every $x \in \mathcal{D}$ there holds

$$\begin{aligned}
\|T_L x\|^2 + \|L_{\vec{\eta}} x\|^2 &= \|T_L x\|^2 + \sum_{i=1}^{N} \eta_i \|L_i x\|^2 \\
&\geq \min_{1 \leq i \leq N} \{\eta_i\} \left(\|T_L x\|^2 + \sum_{i=1}^{N} \|L_i x\|^2 \right)
\end{aligned}$$

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$$= \min_{1\le i\le N}\{\eta_i\}\left(\|T_L x\|^2 + \|Lx\|^2\right)$$
$$\ge \min_{1\le i\le N}\{\eta_i\}\gamma\,\|x\|^2. \qquad \text{(since } L \text{ satisfies (CC))}$$

From this and by virtue of Theorem 2.3 it follows that the family $\{R_{\alpha,\vec{\eta}}\}_{\alpha\in\left(0,\left\|B_{\vec{\eta}}\right\|^2\right)}$ is a regularization for $T^\dagger_{TL_{\vec{\eta}}}$ and therefore for every $y\in\mathcal{D}(T^\dagger_{TL_{\vec{\eta}}})$, $\lim_{\alpha\to0^+} R_{\alpha,\vec{\eta}}\,y = T^\dagger_{TL_{\vec{\eta}}}y$. Moreover, from Theorem 2.3 it also follows that $\lim_{\alpha\to0^+} L_{\vec{\eta}}R_{\alpha,\vec{\eta}}\,y = L_{\vec{\eta}}T^\dagger_{TL_{\vec{\eta}}}y$ and $\lim_{\alpha\to0^+} T_{TL_{\vec{\eta}}}R_{\alpha,\vec{\eta}}\,y = Qy$ (where Q is the orthogonal projection of $\mathcal{Y}$ onto $\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(T_L)}$). □

Remark 3.5. *Note that $x^\dagger_{\vec{\eta}} \doteq T^\dagger_{TL_{\vec{\eta}}}y$ is the best approximate solution of $T_L x = y$ for $x\in\mathcal{D}$, that is, $x^\dagger_{\vec{\eta}}$ is the least-squares solution of the problem $Tx=y$ in $\mathcal{D}$ which satisfies $\left\|x^\dagger_{\vec{\eta}}\right\|_{TL_{\vec{\eta}}} < \|\tilde{x}\|_{TL_{\vec{\eta}}}$ for any other least-squares solution $\tilde{x}$.*

The following result characterizes the regularized solutions $R_{\alpha,\vec{\eta}}\,y$, in the particular case in which the family of functions $\{g_\alpha\}$ is given by $g_\alpha(\lambda)\doteq\frac{1}{\lambda+\alpha}$.

Lemma 3.6. *Let $y\in\mathcal{D}(T^\dagger_{TL_{\vec{\eta}}})$, $L, L_{\vec{\eta}}, T_L, T_0^\dagger, L^\dagger_{TL_{\vec{\eta}}}, T^\dagger_{TL_{\vec{\eta}}}, B_{\vec{\eta}}$ and $R_{\alpha,\vec{\eta}}$ as previously defined and $x_{\alpha,\vec{\eta}} \doteq R_{\alpha,\vec{\eta}}y$ with $g_\alpha(\lambda)\doteq\frac{1}{\lambda+\alpha}$. If the operator L is surjective, then for each fixed $\vec{\alpha}$ ($\vec{\alpha}=\alpha\vec{\eta}$), $x_{\alpha,\vec{\eta}}$ is the unique global minimizer of the generalized Tikhonov-Phillips functional defined by $J_{\vec{\alpha},L_1,L_2,\ldots,L_N}:\mathcal{D}\longrightarrow\mathbb{R}^+$,*

$$J_{\vec{\alpha},L_1,L_2,\ldots,L_N}(x)\doteq\|T_L x - y\|^2 + \sum_{i=1}^N \alpha_i\|L_i x\|^2,$$

i.e.

$$\mathop{arg\,min}_{x\in\mathcal{D}} J_{\vec{\alpha},L_1,L_2,\ldots,L_N}(x) = R_{\alpha,\vec{\eta}}\,y = x_{\alpha,\vec{\eta}}.$$

Proof. Since

$$\|T_L x - y\|^2 + \sum_{i=1}^N \alpha_i\|L_i x\|^2 = \|T_L x - y\|^2 + \sum_{i=1}^N \alpha\eta_i\|L_i x\|^2 = \|T_L x - y\|^2 + \alpha\|L_{\vec{\eta}}x\|^2$$

and the operator $L_{\vec{\eta}}$ is linear, closed and surjective, the lemma follows immediately from Proposition 2.4. □

From Theorem 3.4 and Remark 3.5 we see that if the vector regularization rule $\vec{\alpha}$ is chosen "radially", i.e. $\vec{\alpha}=\alpha\vec{\eta}$ (with $\vec{\eta}\in\mathbb{R}^N_+$ fixed), then the regularized solutions $R_{\alpha,\vec{\eta}}\,y$, with $R_{\alpha,\vec{\eta}}$ defined by (13), converge, as $\alpha\to0^+$, to the least-squares solution of the problem $T_L x = y$ that minimizes $\vec{\eta}\bullet(\|L_1x\|^2,\|L_2x\|^2,\ldots,\|L_Nx\|^2)^T$. Thus, not only convergence is guaranteed but also a characterization of the limiting least-squares solution is obtained. It is also important to note that this characterization depends on the radial rule $\vec{\alpha}=\alpha\vec{\eta}$ only through its direction vector $\vec{\eta}$.

If T is injective and $y\in\mathcal{D}(T^\dagger)$ then there is only one least-squares solution of $Tx=y$ but if T is not injective then there are infinitely many. The choice of the vector $\vec{\alpha}$ is then closely related to the least-squares solution that we are approximating. The choice of the weights α_i play a fundamental role since, once they are chosen, they determine that the least-squares solution which we are approximating is the one that minimizes $\vec{\eta}\bullet(\|L_1x\|^2,\|L_2x\|^2,\ldots,\|L_Nx\|^2)^T$.

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It is also important to point out that without any "*a-priori*" information about properties of the exact solution, it is not clear which nor how many operators L_i one should choose, neither is clear how one should weight them. In some particular cases, however, know properties of the exact solution may provide a hint. In image restoration, for instance, if it is known that the exact solution is "blocky" then it seems reasonable to use a combination of a classical penalizer by taking $L_1 \doteq I$ and one more appropriate for capturing and preserving discontinuities, for instance $L_2 \doteq \nabla$.

3.2 Convergence with differentiable vector regularization rules

In the previous subsection we proved that for each radial direction, given by a unit vector $\vec{\eta}$, of the vector regularization rule $\vec{\alpha}(\alpha) \doteq \alpha\vec{\eta}$, the corresponding regularized solutions converge to the least-squares solution which minimizes $\vec{\eta} \bullet (\|L_1 x\|^2, \|L_2 x\|^2, \ldots, \|L_N x\|^2)^T = \sum_{i=1}^N \eta_i \|L_i x\|^2$. Note in this case that $\frac{d\vec{\alpha}}{d\alpha}(0^+) = \vec{\eta}$. This observation point us to conjecture that it is precisely the direction of the vector regularization rule at $\alpha = 0^+$ (when it exists) what determines the limiting least-squares solution. In the next theorem we shall extend the result of Theorem 3.1 to the case of vector regularization rules which are differentiable at the origin and prove the above conjecture by the affirmative.

Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_N$, $\mathcal{Z}$, $\mathcal{D}$, L, $L_{\vec{\eta}}, T_L, T_0^\dagger, L^\dagger_{TL_{\vec{\eta}}}, T^\dagger_{TL_{\vec{\eta}}}$ and $B_{\vec{\eta}}$, all as defined in the previous section and satisfying the same properties.

Theorem 3.7. *Let $\vec{\alpha}(\alpha) \doteq (\alpha_1(\alpha), \alpha_2(\alpha), \ldots, \alpha_N(\alpha))^T$ be the parameterization of a curve in $\mathbb{R}^N_+$ such that $\vec{\alpha}(\alpha)$ converges to zero as α approaches zero from the right, and assume that there exists the right derivative $\vec{\alpha}'(0^+) = \left.\dfrac{\partial \vec{\alpha}(\alpha)}{\partial \alpha}\right|_{\alpha=0^+}$ of $\vec{\alpha}(\alpha)$ at zero, $\vec{\alpha}'(0^+) \neq 0$ and let $\vec{\eta} \doteq \dfrac{\vec{\alpha}'(0^+)}{\|\vec{\alpha}'(0^+)\|}$. Let $\{g_\alpha\}$ be a spectral regularization method such that*

$$\frac{\partial g_\alpha(\lambda)}{\partial \alpha} \text{ exists for every } \alpha \text{ in a right neighborhood of zero, a.e. for } \lambda \in (0, \infty) \tag{14}$$

and

$$\left|\frac{\partial g_\alpha(\lambda)}{\partial \alpha}\right| = \mathcal{O}\left(\frac{1}{\alpha^2}\right) \text{ uniformly for } \lambda > 0. \tag{15}$$

Define $R_{\vec{\alpha}(\alpha)} : \mathcal{Y} \longrightarrow \mathcal{X}$ as

$$R_{\vec{\alpha}(\alpha)} \doteq T_0^\dagger + L^\dagger_{TL_{\vec{\eta}}} g_{\vec{\alpha}(\alpha)}(B^*_{\vec{\eta}} B_{\vec{\eta}}) B^*_{\vec{\eta}}, \tag{16}$$

where for $z \in \mathcal{Z}$, $g_{\vec{\alpha}(\alpha)}(B^*_{\vec{\eta}} B_{\vec{\eta}})z \doteq \left([g_{\alpha_1(\alpha)}(B^*_{\vec{\eta}} B_{\vec{\eta}})z]_1, [g_{\alpha_2(\alpha)}(B^*_{\vec{\eta}} B_{\vec{\eta}})z]_2, \ldots, [g_{\alpha_N(\alpha)}(B^*_{\vec{\eta}} B_{\vec{\eta}})z]_N\right)^T$. Then $R_{\vec{\alpha}(\alpha)}$ is a family of regularization operators for $T^\dagger_{TL_{\vec{\eta}}}$. Moreover, for every $y \in \mathcal{D}(T^\dagger_{TL_{\vec{\eta}}})$ there holds $\lim_{\alpha \to 0^+} R_{\vec{\alpha}(\alpha)} y = T^\dagger_{TL_{\vec{\eta}}} y$.

Proof. For any fixed $i = 1, 2, \ldots, N$, define $\vec{\gamma} = \vec{\gamma}(\alpha) \doteq \alpha\eta_i \|\vec{\alpha}'(0^+)\| \vec{\eta}$. Clearly $\vec{\gamma}$ is a radial vector regularization rule and $\|\vec{\gamma}\| = \alpha\eta_i \|\vec{\alpha}'(0^+)\|$. Then, from the definitions of $R_{\vec{\alpha}(\alpha)}$ in (16) and $R_{\alpha,\vec{\eta}}$ in (13), and by virtue of Theorem 3.4, one can immediately see that in order to prove this theorem it is sufficient to show that for every $i = 1, 2, \ldots, N$, there holds

$$\left[g_{\vec{\alpha}(\alpha)}(B^*_{\vec{\eta}} B_{\vec{\eta}})z\right]_i - \left[g_{\alpha\eta_i\|\vec{\alpha}'(0^+)\|}(B^*_{\vec{\eta}} B_{\vec{\eta}})z\right]_i \overset{\alpha\to 0^+}{\longrightarrow} 0, \quad \forall\, z \in \mathcal{Z}. \tag{17}$$

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Let then $\{E_\lambda^{B_{\vec{\eta}}B_{\vec{\eta}}^*}\}$ be the spectral family associated to the selfadjoint operator $B_{\vec{\eta}}B_{\vec{\eta}}^*$. Note that for $z \in \mathcal{Z}$ and for any i, $1 \leq i \leq N$, we have that

$$\left\| \left(g_{\alpha_i(\alpha)}(B_{\vec{\eta}}^* B_{\vec{\eta}})z - g_{\alpha\eta_i \|\vec{\alpha}'(0^+)\|}(B_{\vec{\eta}}^* B_{\vec{\eta}})z \right)_i \right\|_{\mathcal{Z}_i}^2 \leq \left\| \int_0^\infty \left(g_{\alpha_i(\alpha)}(\lambda) - g_{\alpha\eta_i \|\vec{\alpha}'(0^+)\|}(\lambda) \right) dE_\lambda^{B_{\vec{\eta}}B_{\vec{\eta}}^*} z \right\|_{\mathcal{Z}}^2$$
$$= \int_0^\infty \left(g_{\alpha_i(\alpha)}(\lambda) - g_{\alpha\eta_i \|\vec{\alpha}'(0^+)\|}(\lambda) \right)^2 d \left\| E_\lambda^{B_{\vec{\eta}}B_{\vec{\eta}}^*} z \right\|_{\mathcal{Z}}^2. \tag{18}$$

On the other hand, since the family of functions $\{g_\alpha\}$ constitutes a spectral regularization method and $\vec{\alpha}(0^+) = 0$, we have that

$$g_{\alpha_i(\alpha)}(\lambda) - g_{\alpha\eta_i \|\vec{\alpha}'(0^+)\|}(\lambda) \overset{\alpha \to 0^+}{\longrightarrow} \frac{1}{\lambda} - \frac{1}{\lambda} = 0, \quad \forall\, \lambda > 0, \forall\, i = 1, 2, \ldots, N \tag{19}$$

and on the other hand, since $\vec{\alpha}(\alpha)$ is differentiable at $\alpha = 0^+$ and $\vec{\eta} = \dfrac{\vec{\alpha}'(0^+)}{\|\vec{\alpha}'(0^+)\|}$, we have that $\vec{\alpha}(\alpha) = \alpha \|\vec{\alpha}'(0^+)\| \, \vec{\eta} + \vec{\beta}(\alpha)$ where $\left\|\vec{\beta}(\alpha)\right\| = \mathcal{O}(\alpha^2)$. Then for every $\alpha > 0$, $\lambda > 0$

$$g_{\alpha_i(\alpha)}(\lambda) - g_{\alpha\eta_i \|\vec{\alpha}'(0^+)\|}(\lambda) = g_{\alpha\eta_i \|\vec{\alpha}'(0^+)\| + \beta_i(\alpha)}(\lambda) - g_{\alpha\eta_i \|\vec{\alpha}'(0^+)\|}(\lambda)$$
$$= \left(\left. \frac{\partial g_\alpha(\lambda)}{\partial \alpha} \right|_{\alpha = \xi_i} \right) \beta_i(\alpha), \text{ (by (14) and the Mean Value Theorem)}$$

(for some $\xi_i \in \mathbb{R}$ between $\alpha\eta_i \|\vec{\alpha}'(0^+)\| + \beta_i(\alpha)$ and $\alpha\eta_i \|\vec{\alpha}'(0^+)\|$). It then follows by virtue of (15) that

$$\forall\, \delta > 0 \text{ (sufficiently small) } \exists\, k < \infty : \left| g_{\alpha_i(\alpha)}(\lambda) - g_{\alpha\eta_i \|\vec{\alpha}'(0^+)\|}(\lambda) \right| \leq k, \ \forall\, \lambda > 0, \forall\, \alpha \in (0, \delta). \tag{20}$$

Finally, (17) follows from (18), (19) and (20) via the Lebesgue Dominated Convergence Theorem. □

4 Applications: image restoration with convex combinations of seminorms

The purpose of this section is to present an application to a simple image restoration problem, of the use of generalized Tikhonov-Phillips methods with penalizers given by linear combinations of squares of seminorms induced by closed operators. The main objective is to show how the choice of penalizers in a generalized Tikhonov-Phillips functional can significantly affect the restored image.

The basic mathematical model for image blurring is given by the following Fredholm integral equation

$$K\, f(x, y) = \int\int_\Omega k(x, y, x', y') f(x', y') dx' dy' = g(x, y), \tag{21}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, $f \in \mathcal{X} \doteq L^2(\Omega)$ represents the original image, k is the so called "point spread function" (PSF) and g is the blurred image. For the examples shown below we used a PSF of "atmospheric turbulence" type, i.e. we chose k to be gaussian:

$$k(x, y, x', y') \ = (2\pi\sigma\tilde{\sigma})^{-1} \exp\left(-\tfrac{1}{2\sigma^2} (x - x')^2 - \tfrac{1}{2\tilde{\sigma}^2} (y - y')^2 \right), \tag{22}$$

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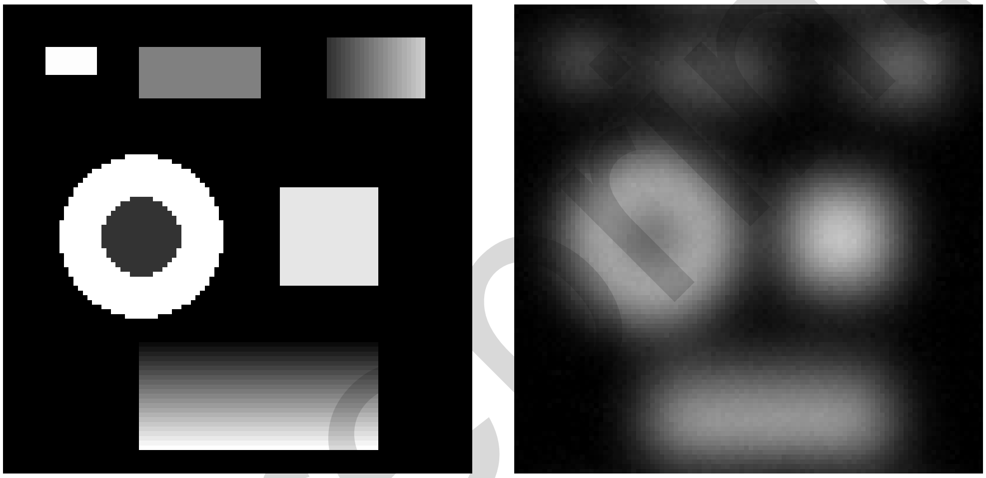
with $\sigma = \tilde{\sigma} = 6$. It is well known ([3]) that with this PSF the operator K in (21) is compact with infinite dimensional range and therefore $K^\dagger$, the Moore-Penrose inverse of K, is unbounded. Generalized Tikhonov-Phillips methods with different penalizers where used to obtain regularized solutions of the problem

$$K\,f = g. \tag{23}$$

For the two numerical examples that follow, problem (23) was discretized in the usual way building the matrix associated to the operator K by imposing periodic boundary conditions (see [6]). The blurred data g was further contaminated with a 1% a gaussian noise (that is with a standard deviation of the order of 1% of $\|g\|_\infty$). Mainly due to computational restrictions, in both cases the size of the images considered is 100×100 pixels.

Example 4.1. Figure 1 shows the original image and the blurred noisy image which constitutes the data for the inverse problem.



(a) (b)

Figure 1: (a) Original image; (b) blurred noisy image.

Six different generalized Tikhonov-Phillips methods with penalizers as in (9) given by

$$W(x) \doteq \alpha \left(w \|L_1 x\|^2 + (1-w)\|L_2 x\|^2 \right) \tag{24}$$

with $0 \leq w \leq 1$, were used to restore f. In all cases the value of the regularization parameter α was computed by means of the L-curve method ([5], [7]).

Figures 2(a) and 2(b) show the restored images obtained with the classical Tikhonov-Phillips methods of orders zero and one, respectively, corresponding to the choices of $w = 1$, $L_1 = I$, $L_2 = \nabla$ and $w = 0$, $L_1 = I$, $L_2 = \nabla$ in (24), respectively.

Figures 3(a)-(d) were obtained using in all cases $L_1 = I$, $L_2 = \nabla$ and four different values of the weight parameter w in (24).

Although some minor differences in the restorations can be observed by simple inspection of the images (measured by the "eyeball norm"), the Improved Signal-to-Noise Ratio (ISNR) defined as

$$ISNR = 10 \log_{10} \left(\frac{\|g - f\|_F^2}{\|f_\alpha - f\|_F^2} \right),$$

(where F denotes the Frobenius norm and f_α is the restored image obtained with regularization parameter α) was computed in order to have an objective parameter to measure and

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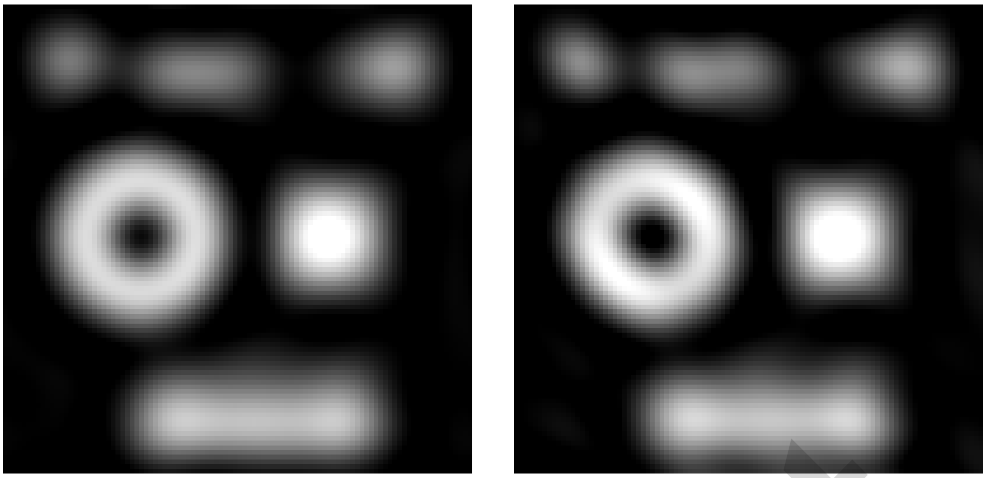

(a) (b)

Figure 2: Restored images; (a) Tikhonov 0 ($w = 1.0$), $\alpha = 0.0167$; (b) Tikhonov 1 ($w = 0.0$), $\alpha = 0.0865$.

compare the quality of all restored images. Table 1 shows the ISNR values corresponding to the six regularization methods used. It is interesting to note that all four combined methods corresponding to non-trivial choices of weight parameters w ($0 < w < 1$), show an improvement in the ISNR value, both in regard to the pure Tikhonov 0 ($w = 1$) and to the pure Tikhonov 1 ($w = 0$) methods.

Method	$w = 1.0$ Fig. 2(a)	$w = 0.0$ Fig. 2(b)	$w = 0.05$ Fig. 3(a)	$w = 0.1$ Fig. 3(b)	$w = 0.2$ Fig. 3(c)	$w = 0.3$ Fig. 3(d)
ISNR (dB)	2.5121	2.6761	2.7464	2.7583	2.7497	2.7325

Table 1: ISNR values of the restored images for Example 4.1

Example 4.2. Figures 4(a)-(b) show the original and degraded image, respectively, for this example, while figures 5(a)-(b) show the restorations obtained with the classical Tikhonov-Phillips methods of order zero and one, respectively. The restorations obtained with the combined methods by using penalizers as in (24) with weight values $w = 0.05$, $w = 0.1$, $w = 0.2$ and $w = 0.3$ are presented in Figures 6(a)-(d). For these six restorations the ISNR values are presented in Table 2. Once again, we observe that the ISNR values of all four non-trivially combined methods are larger than both of those corresponding to the single "pure" methods. The improvements of the combined restorations for this example is even better than those obtained in Example 4.1. It is reasonable to think that this is so due to the fact that although the original image in Example 4.1 is mainly "blocky", the image for Example 4.2 presents both regions of blocky type and regions with nonconstant but regular intensity gradients, for which one could in fact expect that a combined method will do a much better job than any of the pure methods applied separately. Although this can be though of as a purely empirical and somewhat intuitive observation, it points to an important aspect of the theory which deserves further research, namely, that regarding an "optimal" choice of the weight parameters α_i in the functional (9).

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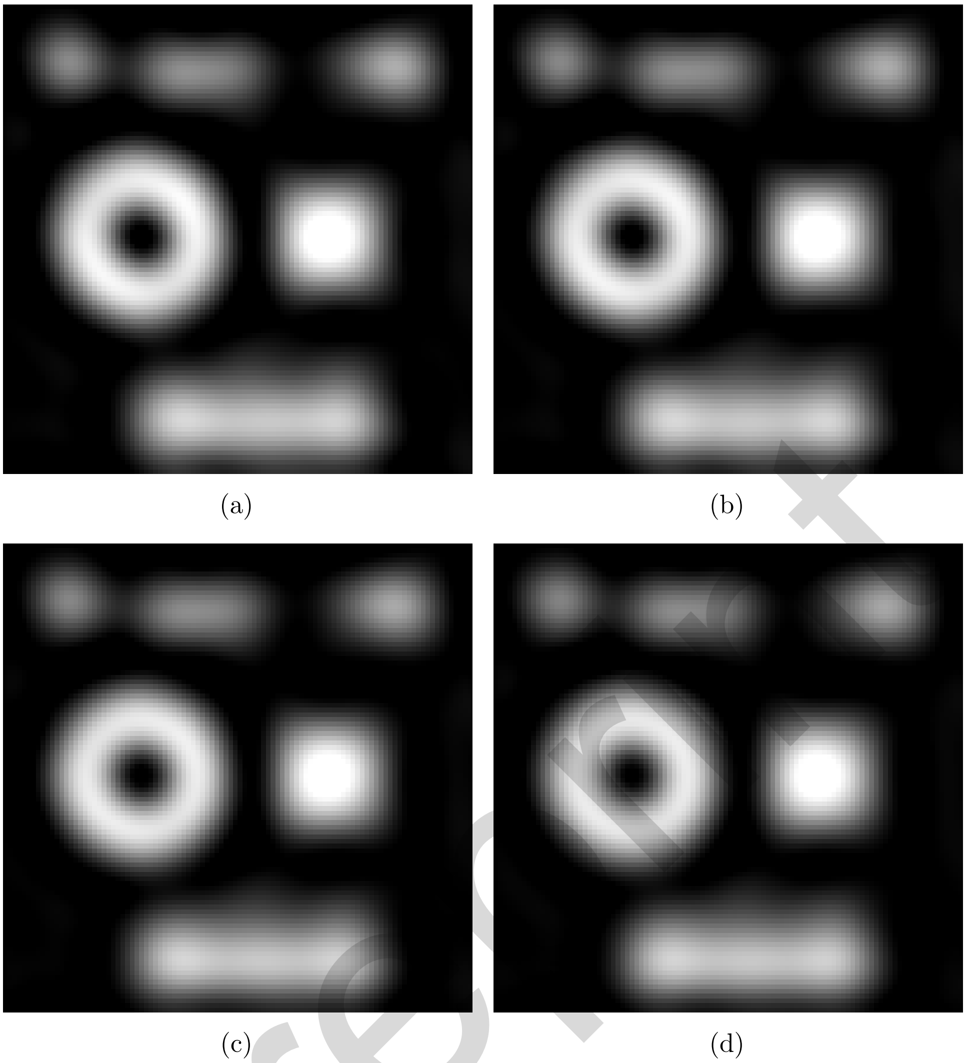

Figure 3: Restorations with combined Tikhonov 0-1 methods; (a) $w = 0.05$, $\alpha = 0.0577$; (b) $w = 0.1$, $\alpha = 0.0463$; (c) $w = 0.2$, $\alpha = 0.0352$; (d) $w = 0.3$, $\alpha = 0.0295$.

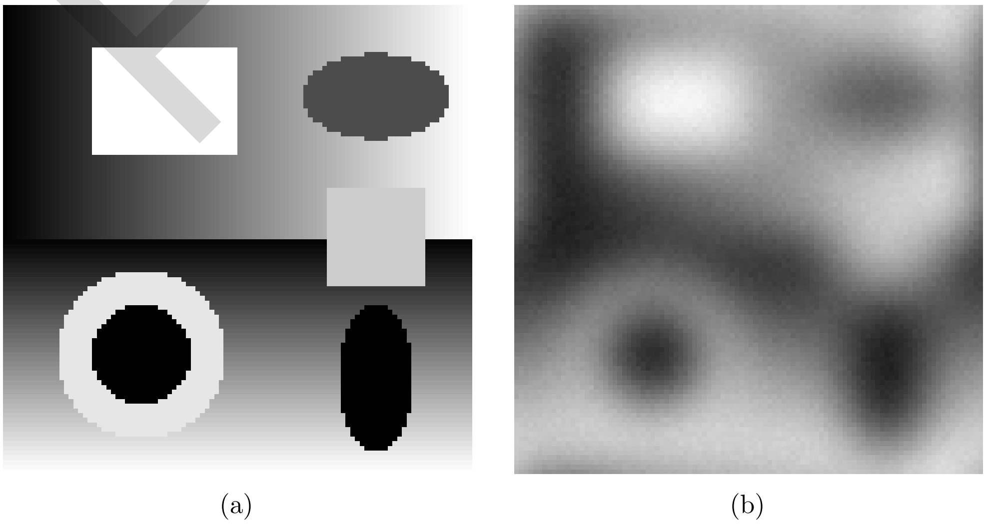

Figure 4: (a) Original image; (b) blurred noisy image.

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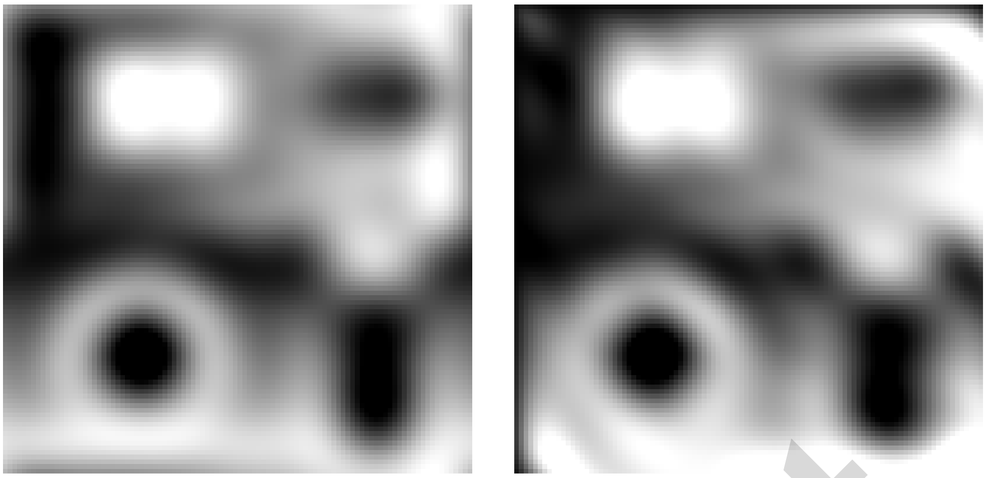
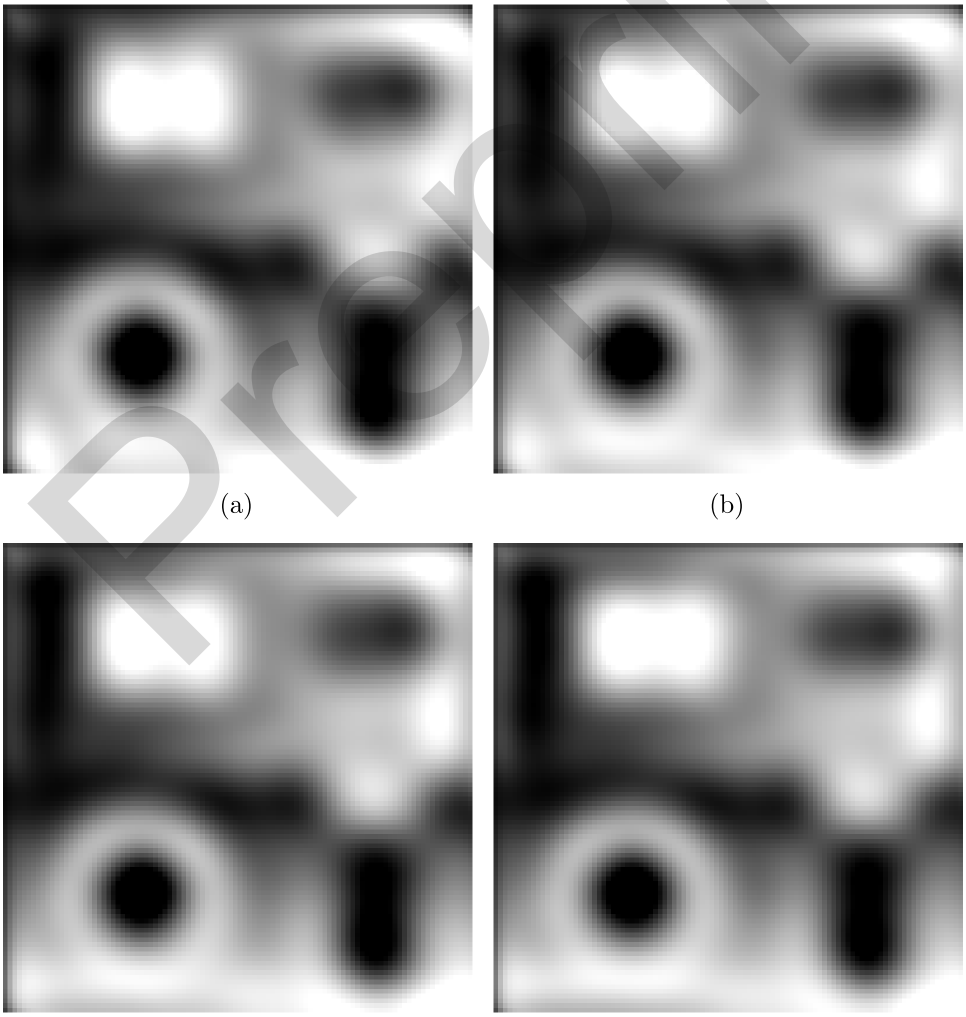


(a) (b)

Figure 5: Restored images; (a) Tikhonov 0 ($w = 1.0$), $\alpha = 0.0121$; (b) Tikhonov 1 ($w = 0.0$), $\alpha = 0.1110$.



(c) (d)

Figure 6: Restorations with combined Tikhonov 0-1 methods; (a) $w = 0.05$, $\alpha = 0.0452$; (b) $w = 0.1$, $\alpha = 0.0338$; (c) $w = 0.2$, $\alpha = 0.0252$; (d) $w = 0.3$, $\alpha = 0.0211$.

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Method	$w = 1.0$ Fig. 5(a)	$w = 0.0$ Fig. 5(b)	$w = 0.05$ Fig. 6(a)	$w = 0.1$ Fig. 6(b)	$w = 0.2$ Fig. 6(c)	$w = 0.3$ Fig. 6(d)
ISNR (dB)	2.2551	2.3776	2.8711	2.9755	2.9643	2.8925

Table 2: ISNR values of the restored images for Example 4.2

5 Conclusions and open issues

In this article we considered regularized solutions of linear inverse ill-posed problems obtained with generalized Tikhonov-Phillips functionals with penalizers given by linear combinations of seminorms induced by closed operators. Convergence of the regularized solutions was proved when the vector regularization rule approaches the origin through appropriate radial and differentiable paths. Characterizations of the limiting solutions were given.

In the previous sections it was proved that when a family of closed operators is used to construct a spectral regularization method as given in (13) or (16), provided that the vector regularization rule is differentiable at the origin, it is the vector $\vec{\eta}$ of relative weights induced by direction of the rule at the origin, what defines the limiting least-squares solution. This is particularly clear for the Tikhonov-Phillips method where the limiting least-squares solution is that which minimizes the convex combination of the squares of the seminorms induced by those closed operators, namely $\vec{\eta} \bullet (\|L_1x\|^2, \|L_2x\|^2, \dots, \|L_Nx\|^2)^T = \sum_{i=1}^N \eta_i \|L_ix\|^2$. Nothing is said nor known, however, about how these weight values η_i (and therefore the limiting direction of the vector regularization rule) should be chosen. Is there an "optimal" value of $\vec{\eta}$ (perhaps measure in terms of the ISNR)? If so, is there any way to explicitly find it? The examples presented in Section 4 show that the quality of the obtained results can greatly depend on the choice of $\vec{\eta}$. This is a problem where more research is needed. Certainly, results in this direction could be of significant relevance in many applied problems.

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