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DYADIC NON LOCAL DIFFUSION. THE POINTWISE CONVERGENCE TO THE INITIAL DATA

MARCELO ACTIS AND HUGO AIMAR

Abstract. In this paper we solve the initial value problem for the diffusion induced by a dyadic fractional derivative in $\mathbb{R}^+$. The main result concerns the pointwise estimate of the maximal operator of the diffusion by the Hardy-Littlewood dyadic maximal operator. As a consequence we obtain the pointwise convergence for the initial data in Lebesgue spaces.

1. Introduction

If $W_t(x)$ denotes the Weierstrass kernel in $\mathbb{R}^n$, the function $u(x,t) = (W_t * u_0)(x)$ solves the heat equation $\frac{\partial u}{\partial t} = \Delta u$ in $\mathbb{R}^{n+1}_+$ and the initial data is attained pointwise provided that u_0 belongs to some $L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$. The main analytical tool involved in the proof of the pointwise convergence is the proof of the boundedness of the $\sup_{t>0} |u(x,t)|$ by the Hardy-Littlewood maximal function.

The above situation can be extended to the case of non local diffusion. In this case the Laplacian in space variables is substituted by the operator $(-\Delta)^{s/2}$, $0 < s < 2$. To be precise, for $0 < s < 2$, the fractional derivative of order s of f is given by the kernel representation of the Dirichlet to Neumann operator [2],

$$D^s f(x) = p.v. \int \frac{f(x) - f(y)}{|x-y|^{n+s}}\, dy.$$

The solution of the diffusion problem associated to D^s,

$$\begin{cases} \frac{\partial u}{\partial t} = D^s u, & \text{in } \mathbb{R}^{n+1}_+, \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^n, \end{cases}$$

for adequate initial data u_0 is provided by the Fourier transform

$$\widehat{u}(\xi, t) = e^{-|\xi|^s t}\widehat{u_0}(\xi).$$

In [1] the authors consider the problem of pointwise convergence to the initial data for a Schrödinger type non local operator associated to the dyadic tilings of $\mathbb{R}^+$ and the Haar system. As it is well known, see for example [3, 5, 4, 6, 8, 7], the pointwise convergence to the initial data for the initial value problem for the Schrödinger operator requires more regularity on u_0 than L^p. In particular, in [1] some kind of Besov regularity for u_0 is involved and a Calderón type sharp maximal operator seems to be natural for that setting.

In this note we aim to consider the diffusion problem associated to the fractional derivative introduced in [1]. In particular we shall prove that the dyadic Hardy-Littlewood maximal function still dominates the situation and that the pointwise convergence to the initial data does not need any regularity. As in the Euclidean case, L^p integrability suffices.

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Let us be precise. Let $\mathscr{D} = \bigcup_{j\in\mathbb{Z}} \mathscr{D}^j$ be the family of all dyadic intervals in $\mathbb{R}^+$. If I belongs to $\mathscr{D}^j$, then $I = I_k^j = [(k-1)2^{-j}, k2^{-j})$ for some $k \in \mathbb{Z}^+$ and $|I| = 2^{-j}$, where the vertical bars denote Lebesgue measure in $\mathbb{R}$.

The family $\mathscr{D}$ is organized in generations: for each $I \in \mathscr{D}^j$ there exists 2 disjoint intervals I^+ and I^- in $\mathscr{D}^{j+1}$ both contained in I, which are precisely the left and right halves of I, respectively. We shall say that I^+ and I^- are "children" of I. An "ancestor" of I is any $J \in \mathscr{D}$ such that $I \subseteq J$. Given I and Q in $\mathscr{D}$, we shall say that J is the "first common ancestor" of them, if J is an ancestor of both I and Q which is contained in every common ancestor of them.

The dyadic distance $\delta(x, y)$ from x to y, both in R^+, is defined as zero when $x = y$ and as the measure of the smallest dyadic interval $J \in \mathscr{D}$ containing both x and y. Notice that for any two points x and y in $\mathbb{R}^+$ $\delta(x, y)$ is well defined since for $|j|$ large enough and j negative the interval $[0, 2^{-j})$ is dyadic and contains x and y. As it is easy to see $|x - y| \le \delta(x, y)$ but $\frac{1}{\delta(x,y)}$ is still singular in the sense that $\int_{\mathbb{R}^+} \frac{dy}{\delta(x,y)} = +\infty$ even when $\int_{(0,1)} \frac{dy}{\delta(x,y)^{1-\epsilon}}$ and $\int_{(1,\infty)} \frac{dy}{\delta(x,y)^{1+\epsilon}}$ are both finite for $\epsilon > 0$. See Lemma 2 in §2.

For $I \in \mathscr{D}$ we shall write h_I to denote the Haar function supported on I. In other words $h_I = |I|^{-\frac{1}{2}}(\chi_{I^-} - \chi_{I^+})$, where χ_E denotes the indicator function of the set E. The system $\{h_I : I \in \mathscr{D}\}$ known as the Haar system is an orthogonal basis for $L^p(\mathbb{R})$ and an unconditional basis for $L^p(\mathbb{R}), 1 < p < \infty$. With $\langle f, h_I\rangle$ we denote the inner product $\int_{\mathbb{R}^+} f h_I dx$ as far as it is well defined. The fractional dyadic derivative of order $\sigma \in (0, 1)$ is defined by

$$\mathcal{D}^\sigma f(x) = \int_{\mathbb{R}^+} \frac{f(x) - f(y)}{\delta(x, y)^{1+\sigma}}\, dy,$$

provided that the integral is absolutely convergent. In this case we say that f is differentiable of order σ in the dyadic sense. Notice that this is the case if for example f is a bounded Lipschitz function in the classical sense, since $|x - y| \le \delta(x, y)$. Later on we shall deal with the Besov classes for which D^σ is well defined. The dyadic Hardy-Littlewood maximal operator is defined for a locally integrable function f defined on $\mathbb{R}^+$ by

$$M_{dy} f(x) = \sup_{x\in I\in\mathscr{D}} \frac{1}{|I|} \int_I |f(y)|\, dy$$

We are now in position to state our main result.

Theorem 1. *Let $0 < \sigma < 1$, $1 \le p \le \infty$ and $u_0 \in L^p(\mathbb{R}^+)$ be given. Then,*

(A) *the function u defined in $\mathbb{R}^+ \times \mathbb{R}^+$ by*

$$u(x, t) = \sum_{I\in\mathscr{D}} e^{-b_\sigma |I|^{-\sigma} t} \langle u_0, h_I\rangle h_I(x)$$

for fixed t is differentiable of order σ in the dyadic sense as a function of x and solves the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{D}^\sigma u, & x \in \mathbb{R}^+, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^+, \end{cases} \tag{1.1}$$

where the initial condition is satisfied in the sense of $L^p(\mathbb{R}^+)$;

(B) *there exists a constant $C > 0$ such that*

$$u^*(x) = \sup_{t>0} |u(x, t)| \le CM_{dy} u_0(x);$$

(C) $\lim_{t\to 0^+} u(x, t) = u_0(x)$ *for almost every $x \in \mathbb{R}^+$.*

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The paper is organized as follows. In Section 2 we obtain the spectral analysis of the operator

$$\mathcal{D}^\sigma f = \int \frac{f(x) - f(y)}{\delta(x,y)^{1+\sigma}} dy$$

in terms of the Haar system. Section 3 is devoted to obtain the maximal estimate contained in statement (B) of Theorem 1. Finally, Section 4 contains the proof of Theorem 1.

2. The dyadic fractional differential operator

The first result in this section is an elementary lemma which reflects the one dimensional character of $\mathbb{R}^+$ equipped with the distance δ.

Lemma 2. *Let $0 < \epsilon < 1$, and let I be a given dyadic interval in $\mathbb{R}^+$. Then, for $x \in I$, we have*

$$\int_I \frac{dy}{\delta(x,y)^{1-\epsilon}} = c_\epsilon |I|^\epsilon$$

and

$$\int_{\mathbb{R}^+ \setminus I} \frac{dy}{\delta(x,y)^{1+\epsilon}} = C_\epsilon |I|^{-\epsilon},$$

where $c_\epsilon = \frac{2^{\epsilon+1}}{2^\epsilon - 1}$ and $C_\epsilon = \frac{1}{2^{\epsilon+1}} \frac{1}{2^\epsilon - 1}$.

Proof. Observe that the ball $B_\delta(x,r)$ is the largest dyadic interval I containing x with length less than r. Then, for $I \in \mathscr{D}^j$ and $x \in I$ we have

$$\begin{aligned}
\int_I \frac{dy}{\delta(x,y)^{1-\epsilon}} &= \int_{B_\delta(x,2^{-j+1})} \frac{dy}{\delta(x,y)^{1-\epsilon}} \\
&= \sum_{k=j-1}^{\infty} \int_{\{y:\, 2^{-k-1} \leq \delta(x,y) < 2^{-k}\}} \frac{dy}{\delta(x,y)^{1-\epsilon}} \\
&= \sum_{k=j-1}^{\infty} |\{y : \delta(x,y) = 2^{-k-1}\}| 2^{-(k+1)(\epsilon-1)} \\
&= 2 \sum_{k=j-1}^{\infty} 2^{-(k+1)\epsilon} = \frac{2^{\epsilon+1}}{2^\epsilon - 1} |I|^\epsilon.
\end{aligned}$$

The proof of the second identity follows the same lines. □

Let us notice that the indicator function of a dyadic interval $I \in \mathscr{D}$ is a Lipschitz function with respect to the distance δ. In fact $|\chi_I(x) - \chi_I(y)| \leq \frac{\delta(x,y)}{|I|}$. Hence for $0 < \sigma < 1$, the integral

$$\int_{\mathbb{R}^+} \frac{\chi_I(x) - \chi_I(y)}{\delta(x,y)^{1+\sigma}} dy$$

is absolutely convergent since for any dyadic interval J we have

$$\begin{aligned}
\int_{\mathbb{R}^+} \frac{\chi_I(x) - \chi_I(y)}{\delta(x,y)^{1+\sigma}} dy &\leq \int_J \frac{\chi_I(x) - \chi_I(y)}{\delta(x,y)^{1+\sigma}} dy + \int_{J^c} \frac{\chi_I(x) - \chi_I(y)}{\delta(x,y)^{1+\sigma}} dy \\
&\leq \frac{1}{|I|} \int_J \frac{1}{\delta(x,y)^\sigma} dy + \int_{J^c} \frac{1}{\delta(x,y)^{1+\sigma}} dy.
\end{aligned}$$

Now, for $0 < \sigma < 1$ we are in position to define the operator $\mathcal{D}^\sigma$ on the linear span $S(\mathcal{H})$ of the Haar system $\mathcal{H}$, which is contained in the linear span of the indicator functions of dyadic intervals, by

$$\mathcal{D}^\sigma f = \int_{\mathbb{R}^+} \frac{f(x) - f(y)}{\delta(x,y)^{1+\sigma}} dy. \tag{2.1}$$

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In [1] the authors prove that Haar functions are the eigenfunctions of $\mathcal{D}^\sigma$. However we will give a simpler alternative proof.

Theorem 3. *Let $\sigma \in \mathbb{R}$ be such that $0 < \sigma < 1$, then for each $h_I \in \mathcal{H}$ we have*

$$\mathcal{D}^\sigma h_I(x) = b_\sigma |I|^{-\sigma} h_I(x), \tag{2.2}$$

with $b_\sigma = 1 + C_\sigma$.

Proof. Notice that for $I, J \in \mathscr{D}$, with $I \cap J = \emptyset$, we have that

$$\delta(x,y) = C, \quad \text{for all } x \in I \text{ and all } y \in J. \tag{2.3}$$

Moreover, the constant $C = |\widetilde{I}|$, where I^0 is the first common ancestor of I and J.

Take $h_I \in \mathcal{H}$. Suppose first that $x \notin I$. Since h_I is supported on I, then $h_I(x) = 0$. Hence

$$\int \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\sigma}} dy = \int_{\mathbb{R}^+ \setminus I} \frac{-h_I(y)}{\delta(x,y)^{1+\sigma}} dy + \int_I \frac{-h_I(y)}{\delta(x,y)^{1+\sigma}} dy,$$

The first integral of the right hand side is zero since $h_I(y) \equiv 0$ for all $y \in \mathbb{R}^+ \setminus I$. For the second integral, since $x \notin I$ and $y \in I$, we apply (2.3) to obtain

$$\int_I \frac{-h_I(y)}{\delta(x,y)^{1+\sigma}} dy = -C(I)^{-1-\sigma} \int_I h_I(y) dy = 0$$

Therefore, we have proved (2.2) for $x \notin I$.

Suppose now that $x \in I$. Let us denote with I^* the child of I which contains x. Then

$$\int_I \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\sigma}} dy = \int_{I^*} \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\sigma}} dy + \int_{I \setminus I^*} \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\sigma}} dy.$$

Since h_I is constant in each child of I, then the integral over I^* is null. Note that in the integral over $I \setminus I^*$ we have $\delta(x,y) = |I|$, then

$$\begin{aligned}
\int_{I \setminus I^*} \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\sigma}} dy &= |I|^{-1-\sigma} \int_{I \setminus I^*} h_I(x) - h_I(y) dy \\
&= |I|^{-1-\sigma} \int_I h_I(x) - h_I(y) dy \\
&= |I|^{-1-\sigma} \left[\int_I h_I(x) dy - \int_I h_I(y) dy \right] \\
&= |I|^{-1-\sigma} h_I(x) |I| \\
&= |I|^{-\sigma} h_I(x). \qquad (2.4)
\end{aligned}$$

Finally, applying Lemma 2, we have that

$$\begin{aligned}
\int_{\mathbb{R}^+ \setminus I} \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\sigma}} dy &= h_I(x) \int_{\mathbb{R}^+ \setminus I} \delta(x,y)^{-1-\sigma} dy \\
&= h_I(x) C_\sigma |I|^{-\sigma}. \qquad (2.5)
\end{aligned}$$

Hence, from (2.4) and (2.5) we obtain

$$\begin{aligned}
\mathcal{D}^\sigma h_I &= \int_I \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\sigma}} dy + \int_{\mathbb{R}^+ \setminus I} \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\sigma}} dy \\
&= |I|^{-\sigma} h_I(x) + C_\sigma |I|^{-\sigma} h_I(x) \\
&= (1 + C_\sigma) |I|^{-\sigma} h_I(x).
\end{aligned}$$

Then we have proved (2.2) for $x \notin I$, and the proof is completed. □

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We want to point out that Theorem 3 allows us to give an alternative definition of $\mathcal{D}^\sigma$. In fact, given $f \in S(\mathcal{H})$ there exists a finite subset $\mathcal{F}_n$ of $\mathscr{D}$ such that

$$f(x) = \sum_{I\in\mathcal{F}_n} \langle f.h_I\rangle\, h_I(x).$$

Then, from the linearity of equation (2.2) we have that

$$\mathcal{D}^\sigma f(x) = \sum_{I\in\mathcal{F}_n} b_\sigma |I|^{-\sigma} \langle f.h_I\rangle\, h_I(x).$$

Notice that the well definition of the above expression follows from the fact that the right hand side is the sum of a finite number of terms. Hence, we can extend $\mathcal{D}^\sigma$ to every $f \in L^p$ in the following way

$$\mathcal{D}^\sigma f(x) = \sum_{I\in\mathscr{D}} b_\sigma |I|^{-\sigma} \langle f.h_I\rangle\, h_I(x), \tag{2.6}$$

provided that the series converges.

3. Maximal function estimates for the solution

The results in Section 2 show that, for $u_0 \in S(\mathcal{H})$, the function

$$u(x,t) := \sum_{I\in\mathfrak{D}} e^{-b_\sigma|I|^{-\sigma}t}\langle u_0, h_I\rangle h_I(x). \tag{3.1}$$

solves the problem

$$\begin{cases} \frac{\partial u}{\partial t} &= \mathcal{D}^\sigma u, \quad x\in\mathbb{R}^+, t>0,\\ u(x,0) &= u_0(x), \quad x\in\mathbb{R}^+,\end{cases}$$

at least formaly. To start with the analysis of the way in which the initial condition is attained, in this section we shall get bounds for the maximal operator associated to $u(x,t)$.

Let us start rewriting as an integral the inner product in 3.1, and changing the integration order to obtain

$$u(x,t) = \int_{\mathbb{R}^+}\left[\sum_{I\in\mathfrak{D}} e^{-b_\sigma|I|^{-\sigma}t} h_I(y)h_I(x)\right] u_0(y)dy.$$

We shall use $k_t(x,y)$ to denote the kernel in the above equation. More precisely,

$$k_t(x,y) = \sum_{I\in\mathfrak{D}} e^{-b_\sigma|I|^{-\sigma}t} h_I(y)h_I(x). \tag{3.2}$$

Then, if K_t denotes the operator with kernel k_t, we have that

$$u(x,t) = \int_{\mathbb{R}^+} k_t(x,y)u_0(y)dy =: K_t u_0(x).$$

The aim of this section is to prove that

$$K^* u_0(x) := \sup_{t>0} |K_t u_0(x)| \le C M_{dy} u_0(x), \tag{3.3}$$

for every $u_0 \in L^p(\mathbb{R}^+)$, where M_{dy} denotes the dyadic Hardy-Littlewood maximal operator. In order to do this, we shall construct a decreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi \in L^1(0,\infty)$ and

$$|k_t(x,y)| = \frac{1}{t^{1/\sigma}}\varphi\left(\frac{\delta(x,y)}{t^{1/\sigma}}\right).$$

Notice first that for fixed x and y in $\mathbb{R}$, only remains in (3.2) the terms in which I contains both x and y. We shall denote I^0 the first common ancestor of x and y,

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and let ℓ be such that $I^0 \in \mathscr{D}^\ell$. Also we shall denote I^j the dyadic interval in $\mathscr{D}^{\ell-j}$ containing I^0. Then

$$\begin{aligned} k_t(x,y) &= \sum_{j\geq 0} e^{-b_\sigma |I^j|^{-\sigma} t} h_{I^j}(y) h_{I^j}(x) \\ &= e^{-b_\sigma |I^0|^{-\sigma} t} h_{I^0}(y) h_{I^0}(x) \\ &\qquad + \sum_{j\geq 1} e^{-b_\sigma |I^j|^{-\sigma} t} h_{I^j}(y) h_{I^j}(x) \end{aligned}$$

Let us observe that, for every $j \geq 1$, x and y belong to the same child of I^j, so that $h_{I^j}(y) = h_{I^j}(x)$. Moreover,

$$h_{I^j}(y) h_{I^j}(x) = \left|I^j\right|^{-1}.$$

Hence,

$$k_t(x,y) = e^{-b_\sigma |I^0|^{-\sigma} t} h_{I^0}(y) h_{I^0}(x) + \sum_{j\geq 1} \frac{e^{-b_\sigma |I^j|^{-\sigma} t}}{|I^j|}.$$

Now, notice that $\delta(x,y) = |I^0|$ and that $|I^j| = 2^j |I^0|$. Also, since x and y belong to different children of I^0, we have that $h_{I^0}(y) h_{I^0}(x) = -|I^0|^{-1}$. Then, we obtain that

$$\begin{aligned} k_t(x,y) &= -e^{-b_\sigma \delta(x,y)^{-\sigma} t} \delta(x,y)^{-1} + \sum_{j\geq 1} \frac{e^{-b_\sigma (2^j \delta(x,y))^{-\sigma} t}}{2^j \delta(x,y)} \\ &= \frac{1}{\delta(x,y)} \left[-e^{-b_\sigma \delta(x,y)^{-\sigma} t} + \sum_{j\geq 1} 2^{-j} e^{-b_\sigma (2^j \delta(x,y))^{-\sigma} t} \right]. \end{aligned}$$

Hence, defining $\varphi : \mathbb{R}^+ \to \mathbb{R}$ as

$$\varphi(s) = \frac{1}{s} \left[-e^{-b_\sigma s^{-\sigma}} + \sum_{j\geq 1} 2^{-j} e^{-b_\sigma (2^j s)^{-\sigma}} \right],$$

we have that

$$k_t(x,y) = \frac{1}{t^{1/\sigma}} \varphi\left(\frac{\delta(x,y)}{t^{1/\sigma}} \right).$$

In order to see that $\varphi \in L^1(\mathbb{R}^+)$, we shall obtain two different bounds for φ. One of them will provide the integrability of φ on $(1,\infty)$, and the other in $[0,1]$. To obtain the first bound, observe first that

$$\varphi(s) \leq \frac{1}{s} \sum_{j\geq 1} 2^{-j} \left[1 - e^{-b_\sigma s^{-\sigma}} \right],$$

which follows easily from the facts that $\sum_{j\geq 1} 2^{-j} = 1$ and that $|e^{-x}| \leq 1$ for $x \in \mathbb{R}^+$. Then, from the Taylor series for the exponential function we obtain

$$\varphi(s) \leq \frac{1}{s} \sum_{j\geq 1} 2^{-j} \left[\frac{b_\sigma}{s^\sigma} \right] = \frac{b_\sigma}{s^{1+\sigma}},$$

that give us the integrability of φ on $(1,\infty)$.

Finally, notice that

$$\varphi(s) \leq \frac{1}{s} \left[e^{-b_\sigma s^{-\sigma}} + \sum_{j\geq 1} 2^{-j} e^{-b_\sigma (2^j s)^{-\sigma}} \right]$$

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$$\leq \frac{1}{s}\left[e^{-b_\sigma s^{-\sigma}} + \sum_{j\geq 1} 2^{-j} e^{-b_\sigma s^{-\sigma}}\right]$$
$$\leq \frac{2e^{-b_\sigma s^{-\sigma}}}{s}.$$

The above inequality implies that $\varphi \in L^\infty(\mathbb{R}^+)$, and therefore φ is locally integrable.

Hence,

$$\begin{aligned}
|K_t u_0(x)| &\leq \int_{\mathbb{R}^+} |k_t(x,y)||u_0(y)|\,dy \\
&= \int_{\mathbb{R}^+} \frac{1}{t^{1/\sigma}}\varphi\left(\frac{\delta(x,y)}{t^{1/\sigma}}\right)|u_0(y)|\,dy \\
&= \sum_{j=-\infty}^{\infty} \frac{1}{t^{1/\sigma}} \int_{\{y: t^{1/\sigma}2^j \leq \delta(x,y) < t^{1/\sigma}2^{j+1}\}} \varphi\left(\frac{\delta(x,y)}{t^{1/\sigma}}\right)|u_0(y)|\,dy \\
&\leq \sum_{j=-\infty}^{\infty} 2^{j+1}\varphi(2^j)\frac{1}{t^{1/\sigma}2^{j+1}} \int_{B_\delta(x, t^{1/\sigma}2^{j+1})} |u_0(y)|\,dy.
\end{aligned}$$

Since $|B_\delta(x,r)| < r$ and each B_δ is a dyadic interval, we have

$$\begin{aligned}
|K_t u_0(x)| &\leq \sum_{j=-\infty}^{\infty} 2^{j+1}\varphi(2^j)\frac{1}{|B_\delta(x, t^{1/\sigma}2^{j+1})|} \int_{B_\delta(x, t^{1/\sigma}2^{j+1})} |u_0(y)|\,dy \\
&\leq \sum_{j=-\infty}^{\infty} 2^{j+1}\varphi(2^j) M_{dy} u_0(x) \\
&= 4M_{dy}u_0(x) \sum_{j=-\infty}^{\infty} \int_{\{y: 2^{j-1} \leq y < 2^j\}} \varphi(2^j)\,dy \\
&\leq 4M_{dy}u_0(x) \int_{\mathbb{R}^+} \varphi(y)\,dy, \\
&\leq 4\|\varphi\|_{L^1} M_{dy} u_0(x).
\end{aligned}$$

Therefore, taking supremum in t we obtain

$$\sup_{t>0} |K_t u_0(x)| \leq 4\|\varphi\|_{L^1} M_{dy} u_0(x),$$

which completes the proof of (3.3).

4. Proof of Theorem 1

Proof of (A). Let us start by noticing that if $a = \{a_I\}_{I\in\mathscr{D}}$ is a bounded sequence of scalars then, from the equivalence of the L^p norm of f and the L^p norm of its square function $S(f) = \left(\sum_{I\in\mathscr{D}} |\langle f, h_I\rangle|^2 |h_I|^2\right)^{\frac{1}{2}}$, the operator

$$T_a f(x) = \sum_{I\in\mathscr{D}} a_I \langle f, h_I\rangle h_I$$

is bounded in L^p with $\|T_a\| \leq C\|a\|_{\ell^\infty} = C\sup_{I\in\mathscr{D}} |a_I|$.

For $t > 0$ fixed the sequence $\{e^{-b_\sigma |I|^{-\sigma} t}\}$ is bounded, hence $u(x,t)$ belongs to L^p as a function of x and $\|u\|_{L^p} \leq C\|u_0\|_{L^p}$. Also, for fixed $t > 0$,

$$\mathcal{D}^\sigma u(x,t) = \sum_{I\in\mathscr{D}} b_\sigma |I|^{-\sigma} \langle u, h_I\rangle\, h_I(x)$$

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$$= \sum_{I\in\mathscr{D}} b_\sigma |I|^{-\sigma} e^{-b_\sigma |I|^{-\sigma} t} \langle u_0, h_I\rangle h_I(x).$$

belongs to L^p as a function of x, since $b_\sigma |I|^{-\sigma} e^{-b_\sigma |I|^{-\sigma} t} \leq \frac{1}{te}$. Morover $\|\mathcal{D}^\sigma u\|_{L^p} \leq \frac{C}{t}\|u_0\|_{L^p}$.

To prove that the differential equation in (1.1) holds, let us start showing that for $t>0$ fixed

$$\sup_{I\in\mathscr{D}} \left| \frac{e^{-b_\sigma |I|^{-\sigma}(t+h)} - e^{-b_\sigma |I|^{-\sigma} t}}{h} + b_\sigma |I|^{-\sigma} e^{-b_\sigma |I|^{-\sigma} t} \right| \longrightarrow 0, \tag{4.1}$$

when $h\to 0$. This is equivalent to

$$\sup_{I\in\mathscr{D}} \left| \frac{e^{-b_\sigma |I|^{-\sigma} t}}{h} \left[e^{-b_\sigma |I|^{-\sigma} h} - 1 + b_\sigma |I|^{-\sigma} h \right] \right| \longrightarrow 0,$$

when $h\to 0$. Using the Taylor's series of the exponential function we have that

$$\begin{aligned} &\left| \frac{e^{-b_\sigma |I|^{-\sigma} t}}{h} \left[e^{-b_\sigma |I|^{-\sigma} h} - 1 + b_\sigma |I|^{-\sigma} h \right] \right| \\ &\qquad \leq \left| \frac{e^{-b_\sigma |I|^{-\sigma} t}}{h} \left[h^2 \max_{0\leq s\leq h} \left| (b_\sigma |I|^{-\sigma})^2 e^{-b_\sigma |I|^{-\sigma} s} \right| \right] \right| \\ &\qquad = \left| \frac{b_\sigma^2}{|I|^{-2\sigma}} e^{-b_\sigma |I|^{-\sigma} t} h \right| \\ &\qquad \leq \left| \frac{b_\sigma^2}{|I|^{-2\sigma}} e^{-b_\sigma |I|^{-\sigma} t} \right| |h| . \end{aligned}$$

Hence, to obtain (4.1) it suffices to see that

$$\sup_{I\in\mathscr{D}} \left| \frac{b_\sigma^2}{|I|^{-2\sigma}} e^{-b_\sigma |I|^{-\sigma} t} \right| < \infty.$$

Since

$$\left| \frac{b_\sigma^2}{|I|^{-2\sigma}} e^{-b_\sigma t |I|^{-\sigma}} \right| \leq 4(te)^{-2},$$

the first equation of (1.1) holds.

Finally, to prove the pointwise convergence to the initial data in L^p, i.e.

$$u(x,t) \xrightarrow{L^p} u_0(x), \qquad \text{cuando } t\to 0, \tag{4.2}$$

we need to proceed in a different way since for every fixed $t>0$

$$\sup_{I\in\mathscr{D}} \left| e^{-b_\sigma |I|^{-\sigma} t} - 1 \right| = 1$$

However, we will use the fact that for every $F\in L^p$ the projection operator

$$P_i f = \sum_{j<i} \sum_{I\in\mathscr{D}^j} \langle f, h_I\rangle h_I$$

converges to f in L^p when i tends to infinity, or equivalently,

$$\sum_{j\geq i} \sum_{I\in\mathscr{D}^j} \langle f, h_I\rangle h_I \xrightarrow{L^p} 0,$$

when i tends to infinity. For a fixed $\epsilon>0$, let us choose ℓ large enough such that

$$\left\| \left(\sum_{j>\ell} \sum_{I\in\mathscr{D}^j} |\langle u_0, h_I\rangle|^2 |h_I|^2 \right)^{\frac{1}{2}} \right\|_{L^p} < \epsilon. \tag{4.3}$$

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Observe that for every $I \in \mathscr{D}^j$ with $j \leq \ell$ we have that $|I| \leq 2^{-\ell}$, so we can choose t_0 small enough such that

$$|e^{-b_\sigma |I|^{-\sigma} t} - 1| = 1 - e^{-b_\sigma |I|^{-\sigma} t} \leq 1 - e^{-b_\sigma 2^{\ell\sigma} t} < \epsilon, \tag{4.4}$$

for every $t < t_0$. Now, observe that

$$\begin{aligned}\|u - u_0\|_{L^p} &\precsim \left\| \left(\sum_{I \in \mathscr{D}} |e^{-b_\sigma |I|^{-\sigma} t} - 1| |\langle u_0, h_I \rangle|^2 |h_I|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \\ &\leq \left\| \left(\sum_{j \leq \ell} \sum_{I \in \mathscr{D}^j} |e^{-b_\sigma |I|^{-\sigma} t} - 1| |\langle u_0, h_I \rangle|^2 |h_I|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \\ &\quad + \left\| \left(\sum_{j > \ell} \sum_{I \in \mathscr{D}^j} |e^{-b_\sigma |I|^{-\sigma} t} - 1| |\langle u_0, h_I \rangle|^2 |h_I|^2 \right)^{\frac{1}{2}} \right\|_{L^p}.\end{aligned}$$

Therefore, from (4.3) and (4.4) we obtain

$$\begin{aligned}\|u - u_0\|_{L^p} &\precsim \epsilon \left\| \left(\sum_{j \leq \ell} \sum_{I \in \mathscr{D}^j} |\langle u_0, h_I \rangle|^2 |h_I|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \\ &\quad + 2 \left\| \left(\sum_{j > \ell} \sum_{I \in \mathscr{D}^j} |\langle u_0, h_I \rangle|^2 |h_I|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \\ &\precsim \epsilon \|u_0\|_{L^p} + 2\epsilon,\end{aligned}$$

then (4.2) holds and the proof of (A) is complete.

Proof of (B). This part of the theorem has already been proved in section 3 in the proof of the estimate (3.3).

Proof of (C). The pointwise convergence to the initial data, as usual, is an immediate consequence of the boundedness on L^p of the maximal operator u^* and the pointwise convergence in a dense subset of L^p. We will sketch a brief proof for sake of completeness.

Since we already know that $K_t f \to f$ in the L^p sense as $t \to 0^+$, in order to prove the pointwise convergence, define

$$E = \{ f \in L^p : \lim_{t \to 0^+} K_t f \text{ exists for almost every } x \in \mathbb{R}^+ \}.$$

Notice that $S(\mathcal{H}) \subseteq E \subseteq L^p$. Since $S(\mathcal{H})$ is dense in L^p, then we only need to prove that E is a closed subset of L^p. Let $\{f_n\}$ be a sequence contained in E such that f_n converges in L^p to a function f. To see that $f \in E$ it is enough to prove that for all $\epsilon > 0$ we have

$$|E_\epsilon| := \left| \left\{ x : \limsup_{t \to 0^+} K_t f(x) - \liminf_{t \to 0^+} K_t f(x) > \epsilon \right\} \right| = 0. \tag{4.5}$$

For every n we can write

$$\begin{aligned}|E_\epsilon| \leq & \left| \left\{ x : \limsup_{t \to 0^+} K_t f_n(x) - \liminf_{t \to 0^+} K_t f_n(x) > \frac{\epsilon}{3} \right\} \right| \\ & + \left| \left\{ x : \limsup_{t \to 0^+} K_t (f_n - f)(x) > \frac{\epsilon}{3} \right\} \right| + \left| \left\{ x : \liminf_{t \to 0^+} K_t (f_n - f)(x) > \frac{\epsilon}{3} \right\} \right|.\end{aligned}$$

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The first term is zero since $f_n \in E$. For the other two terms we will use the boundedness on L^p of the maximal operator K^* which follows from the item (B). Notice that for every function g we have that

$$\left| \limsup_{t\to 0^+} K_t g(x) \right| \leq K^* g(x).$$

Then, since K^* is bounded on L^p and therefore weakly bounded on L^p, we obtain

$$\left| \left\{ x : \limsup_{t\to 0^+} K_t(f_n - f)(x) > \frac{\epsilon}{3} \right\} \right| \precsim \frac{1}{\epsilon^p} \|f_n - f\|_{L^p}.$$

Similarly we can show that

$$\left| \left\{ x : \liminf_{t\to 0^+} K_t(f_n - f)(x) > \frac{\epsilon}{3} \right\} \right| \precsim \frac{1}{\epsilon^p} \|f_n - f\|_{L^p}.$$

Hence,

$$|E_\epsilon| \precsim \frac{1}{\epsilon^p} \|f_n - f\|_{L^p}.$$

When n tends to infinity we have (4.5). Then E is closed and therefore $E = L^p$. This means that for every $u_0 \in L^p$ we have that

$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} K_t u_0 \quad \text{exists.}$$

But we already know that $u(x,t) \to u_0(x)$ when $t \to 0^+$ in L^p, then (C) follows, which completes the proof. □

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