The logic of rational polyhedra

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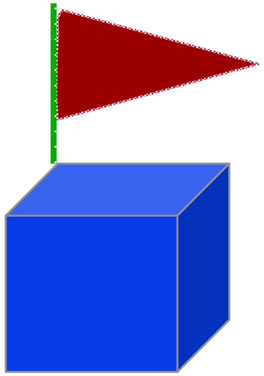
a polyhedron is a finite union P of simplexes S_i in $\mathbf{R}^n$



P need not be convex

P need not be connected

P may have parts of different dimensions

a polyhedron $P = \cup S_i$ is said to be **rational**
if so are the vertices of every simplex S_i

Erlangen geometry of a group of transformations

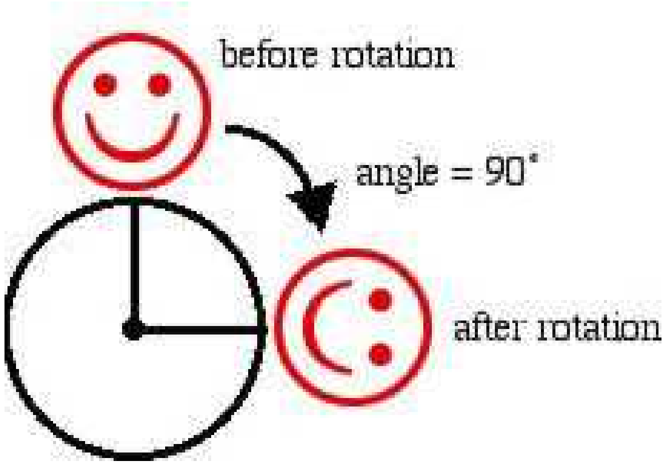
every group G of transformations in $\mathbf{R}^n$ generates a geometry

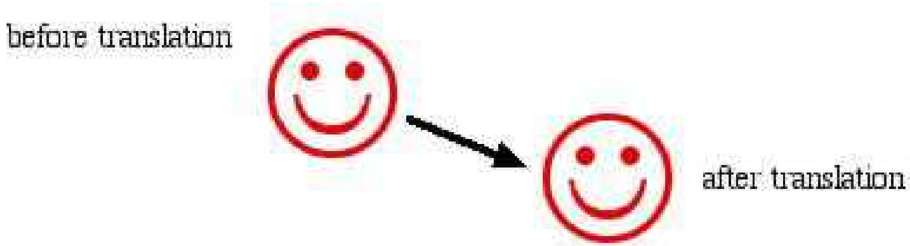
EXAMPLE: E_n = the **euclidean group** of affinities in $\mathbf{R}^n$
A typical element of E_n is a map of the form $x \longrightarrow O_n x + t$
where O is an orthogonal nxn matrix, and t is in $\mathbf{R}^n$

E_n is the *semidirect product* of the orthogonal group O_n and $\mathbf{R}^n$

we are all familiar with E_n-invariant measures:

Lebesgue measure is invariant under the euclidean group E_n





as a consequence of additivity, Lebesgue measure is **invariant** under piecewise linear 1-1 maps h, provided each linear piece of h is given by some element of E_n

another geometry for rational polyhedra in $\mathbf{R}^n$

for each n=1,2,..., let us consider the geometry arising from the group G_n of affine maps in $\mathbf{R}^n$ of the form

$$x \longrightarrow Ux + t$$

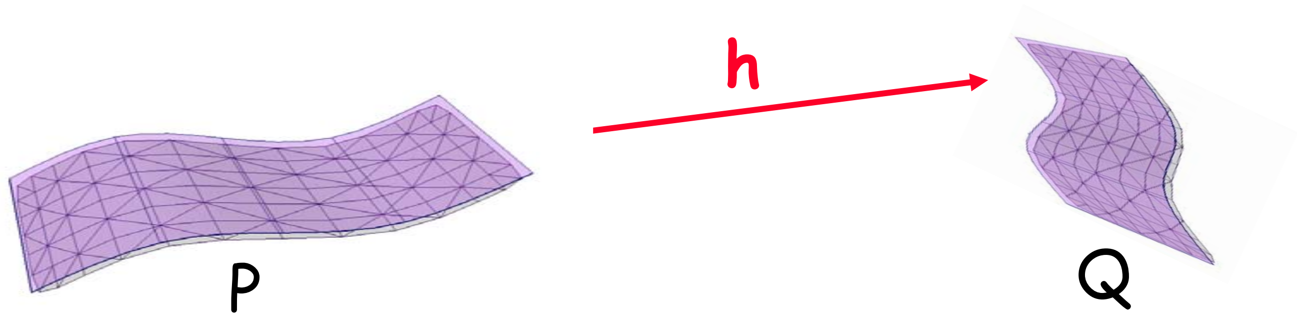
where U is an integer nxn matrix with determinant ± 1, and t is an integer vector in $\mathbf{Z}^n$

G_n is known as the *semidirect product* of the unimodular group $GL(n,\mathbf{Z}) = aut(\mathbf{Z}^n)$ and the translation group $\mathbf{Z}^n$

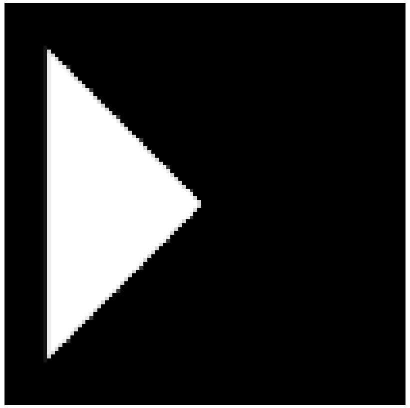
we will construct G_n-invariant measures.
By additivity, these are automatically invariant under piecewise linear 1-1 maps where each piece belongs to G_n

Z-homeomorphism =
PL-homeomorphism **with integer coefficients**

DEFINITION Two rational polyhedra P and Q are **Z-homeomorphic** if there is a PL-homeomorphism h of P onto Q such that every piece of h as well as of of its inverse h^{-1} has integer coefficients

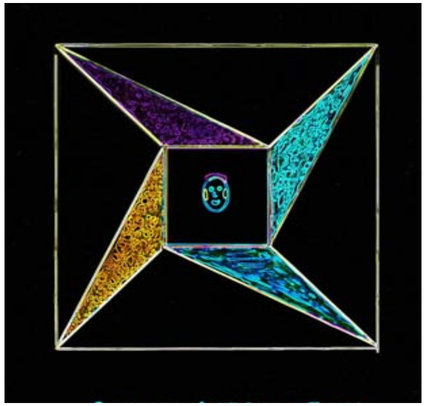


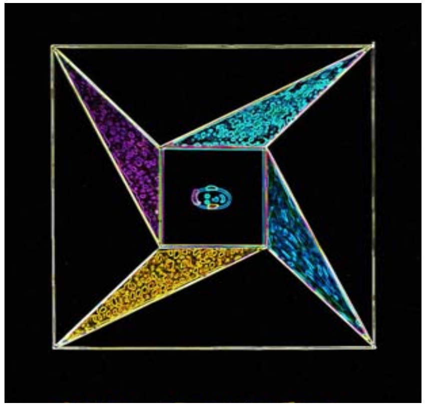
the action of piecewise G_n-linear maps



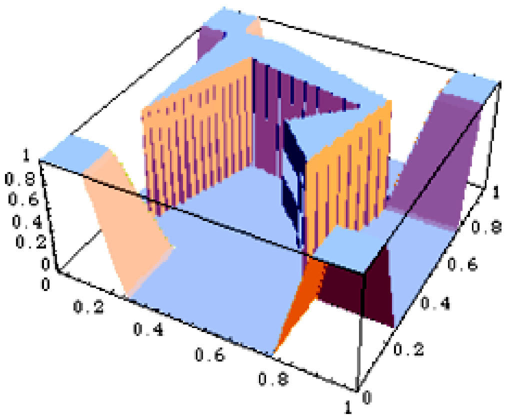
before

after





motivation: why **logic**?



rational polyhedra are the **affine varieties** of the "polynomials" given by formulas in a certain logic L_∞

Z-homeomorphism corresponds to isomorphism in the algebras of L_∞

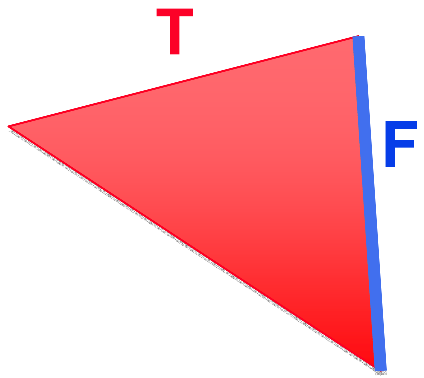
just as homeomorphism corresponds to isomorphism in the algebras of boolean logic (by Stone duality theorem)

Z-homeomorphism does not preserve the usual measure of rational polyhedra P in $\mathbf{R}^n$ when $\dim(P) < n$

these two black segments are **Z**-homeomorphic, but their lengths are different to construct an invariant measure of rational polyhedra in the geometry of the group G_n we need the following fundamental notion (taken from algebraic geometry)

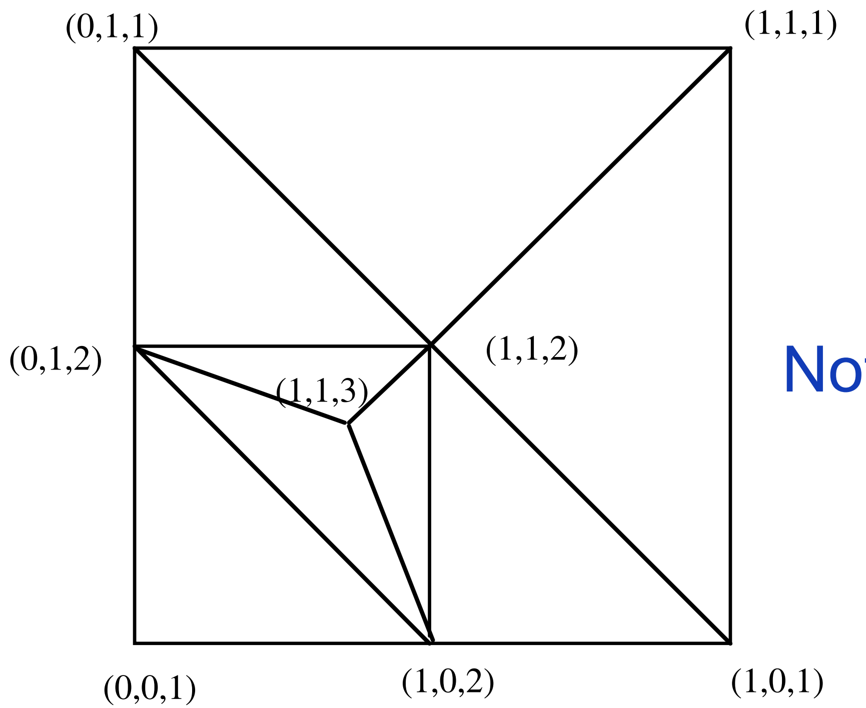
regularity of a simplex T = conv(v_0,...,v_n)

DEFINITION The **denominator** d=den(x) of a rational point x is the least common denominator of the coordinates of x



DEFINITION A simplex **T** is **regular** if it is rational, and for each face **F** of **T**, each rational point in the interior of **F** has a denominator ≥ the sum of the denominators of the vertices of **F**

regular triangulation of a rational polyhedron



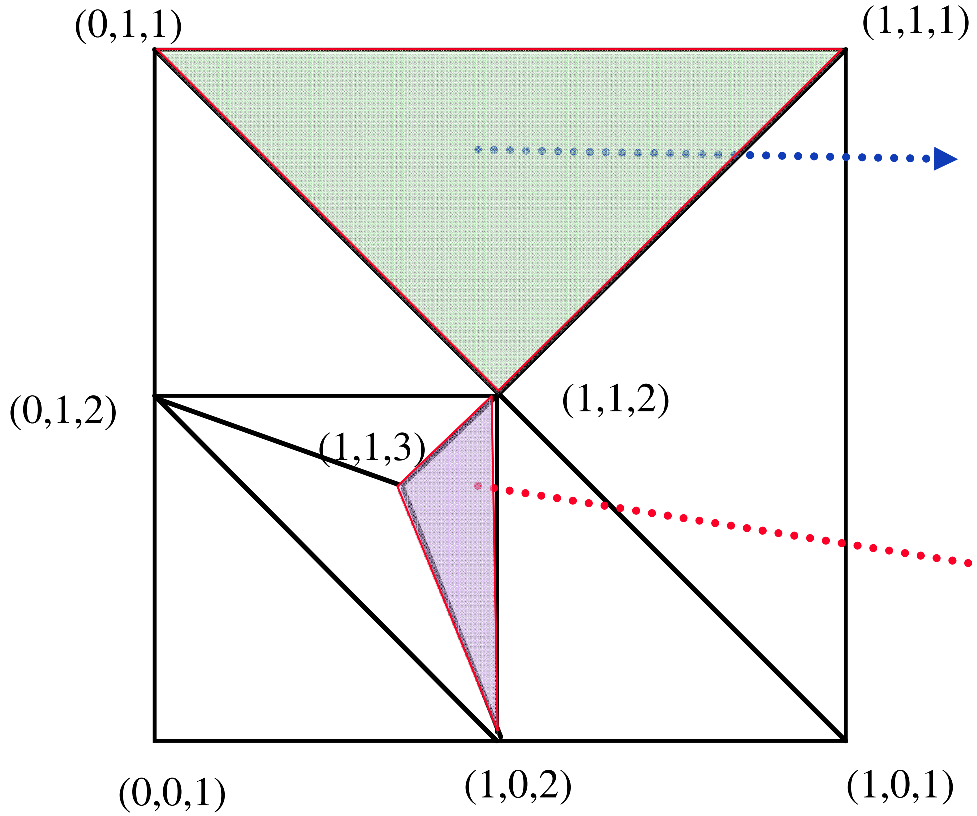
all its simplexes are regular

Note: each vertex (x/d, y/d) is written in homogeneous form, (x, y, d)

Minkowski proved: *The regularity of a simplex T means that the matrix of the homogeneous coordinates of the vertices of T is (extendible to) a unimodular integer matrix*

volume of a regular simplex $T = \mathrm{conv}(v_0,\ldots,v_n)$

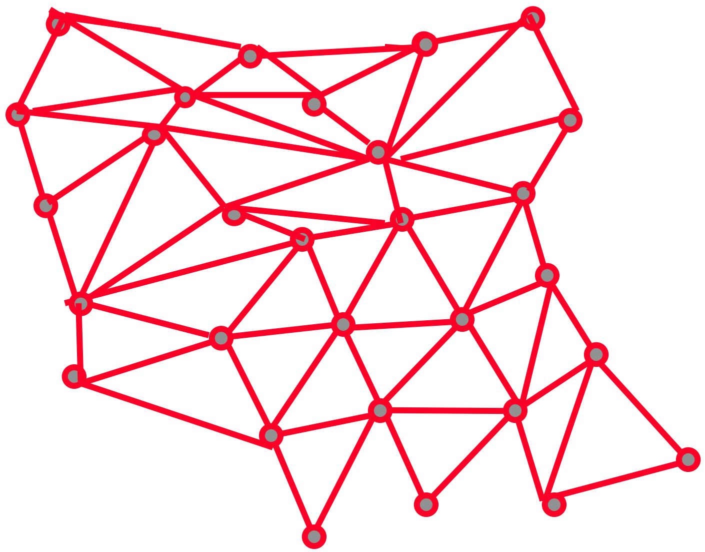
$$\mathrm{vol}(T) = (n!\ \mathrm{den}(v_0)\cdots\mathrm{den}(v_n))^{-1}$$



1/(2x1x1x2)=1/4= the Lebesgue area of the triangle

1/(2x3x2x2)=1/24 also coinciding with its Lebesgue area...

the volume of an arbitrary rational polyhedron P (equipped with a regular triangulation Δ)



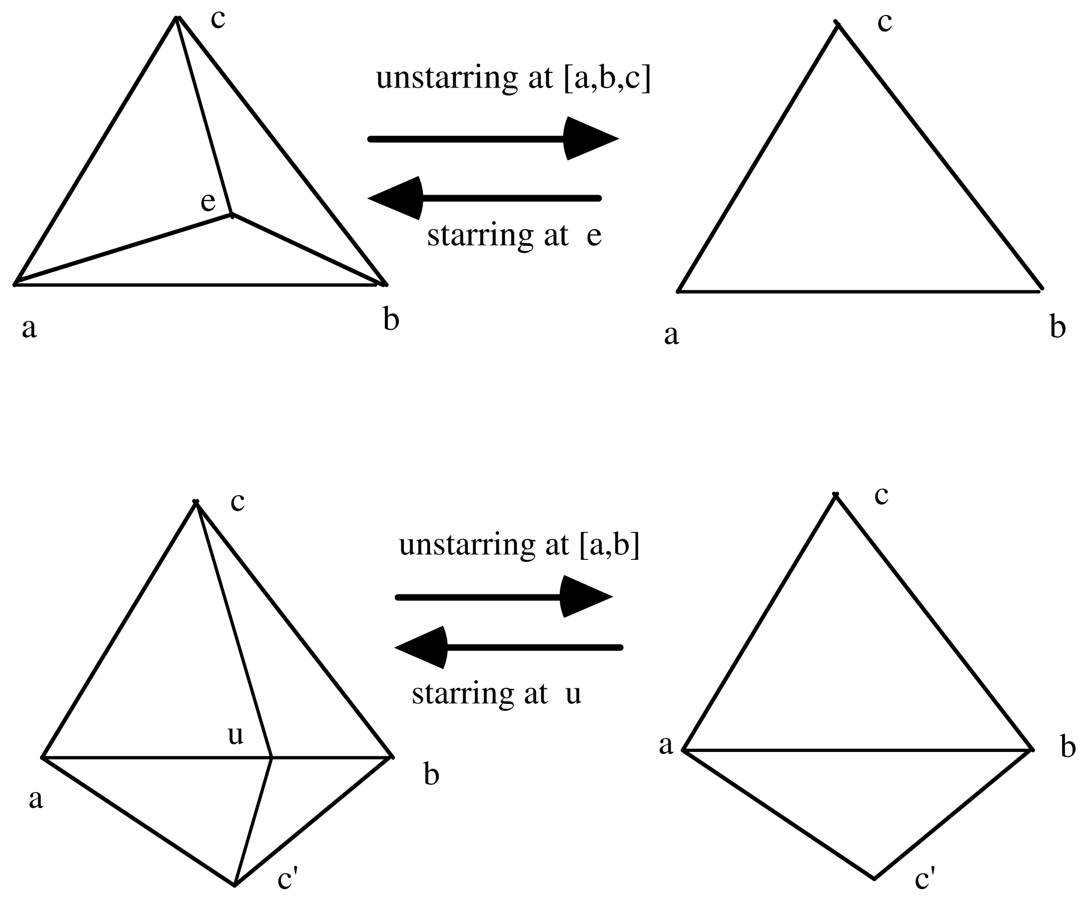
We first calculate the volume of each simplex dΔ of maximum dimension.

Then we set

$$\mathrm{Vol}(P) = \sum \mathrm{Vol}(d\Delta)$$

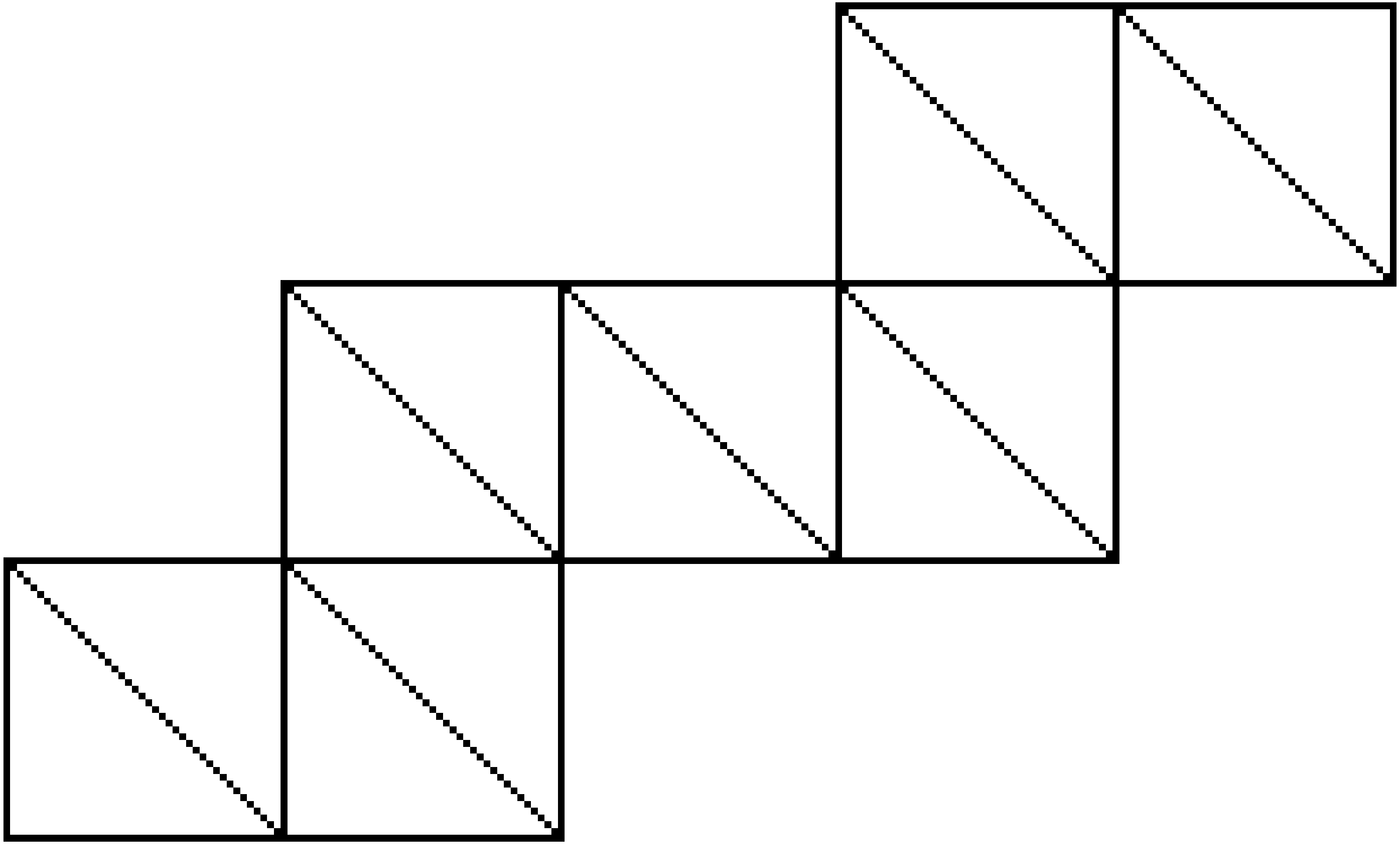
to show that all this makes mathematical sense, we need a couple of results from **toric varieties**

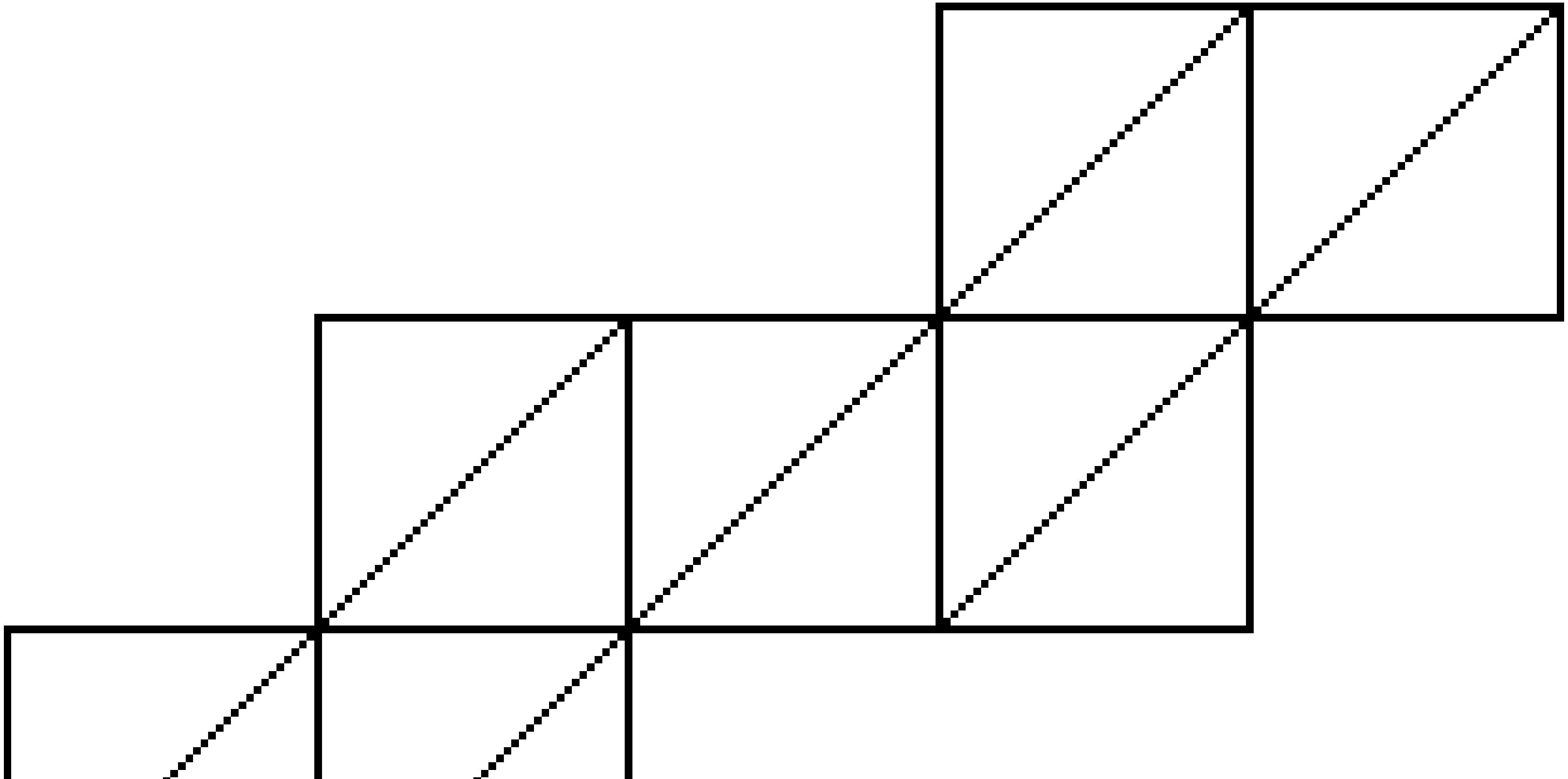
dynamics of regular triangulations

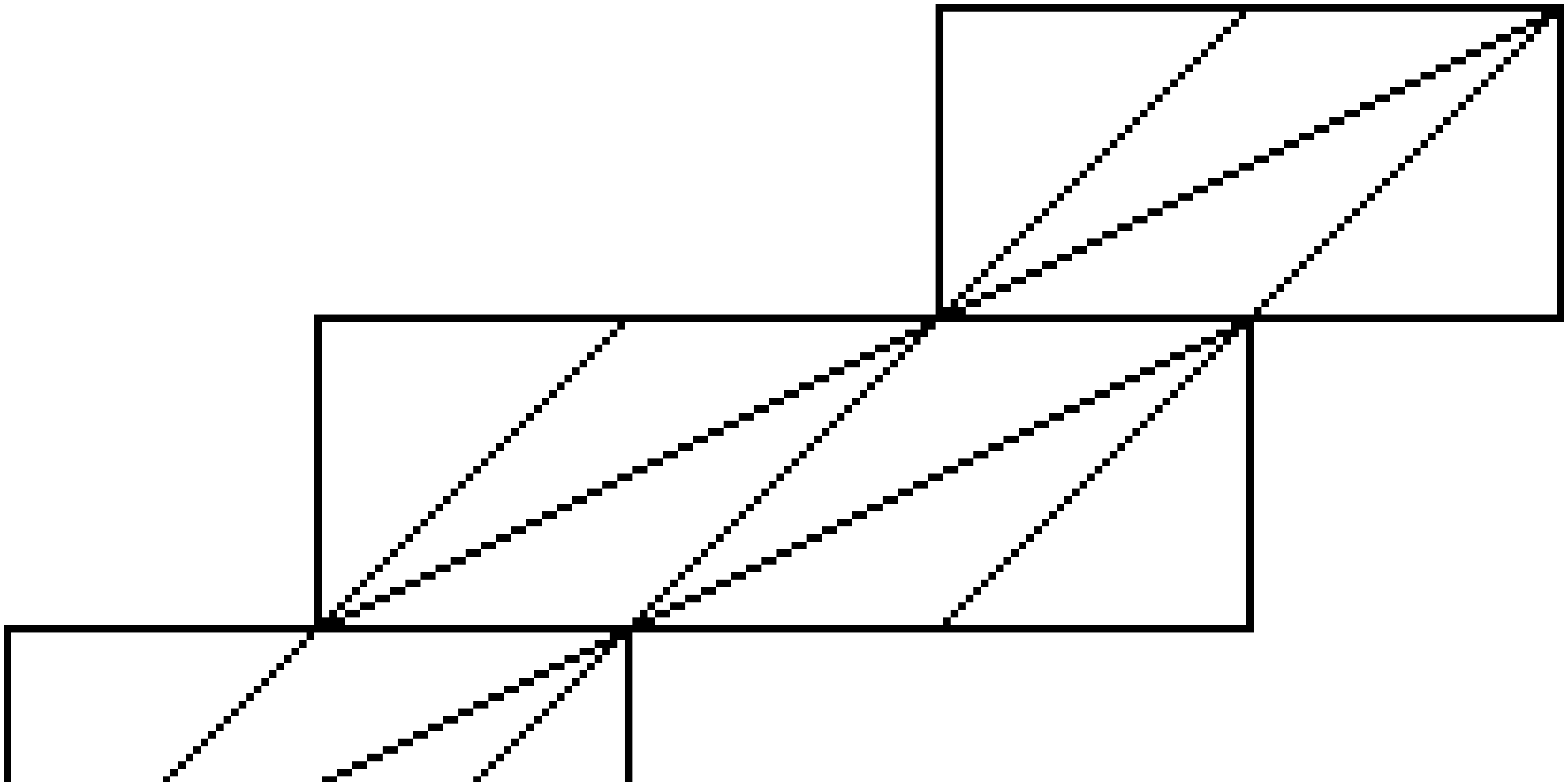


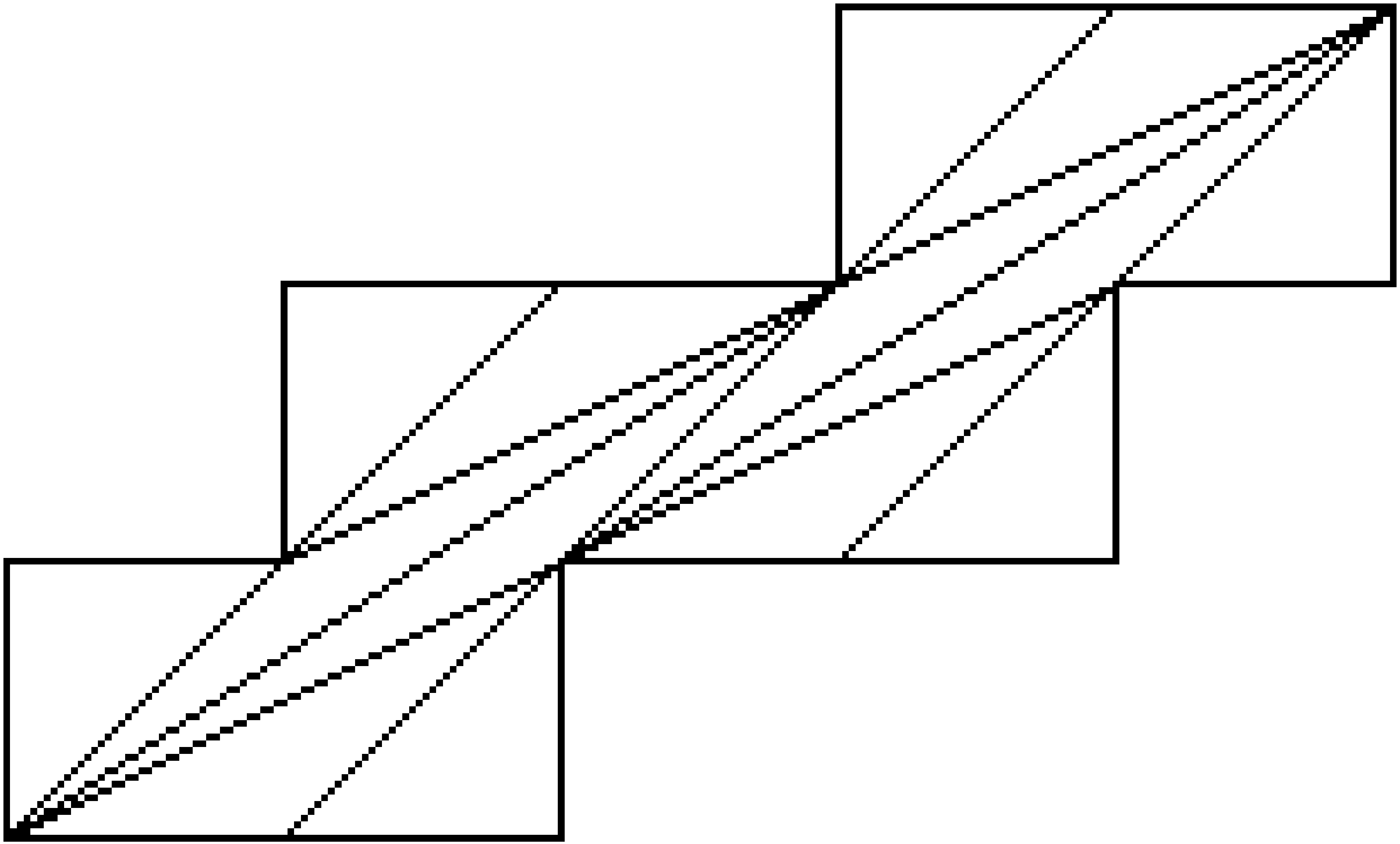
a first main result
(the solution of the weak Oda conjecture by Wlodarczyk-Morelli)

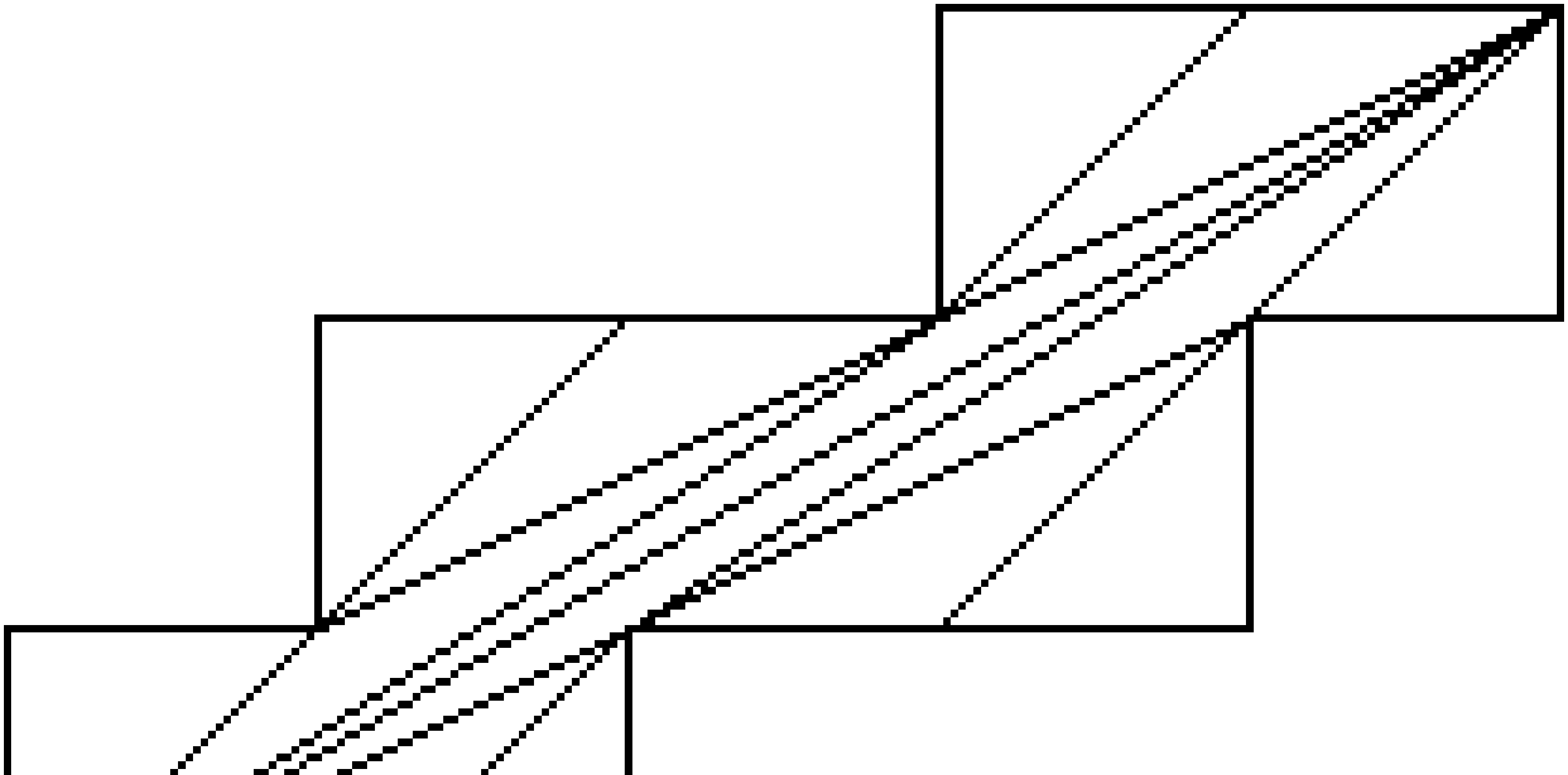
THEOREM *Any two regular triangulations of the same rational polyhedron are connected by a path of blow-ups and blow-downs.*







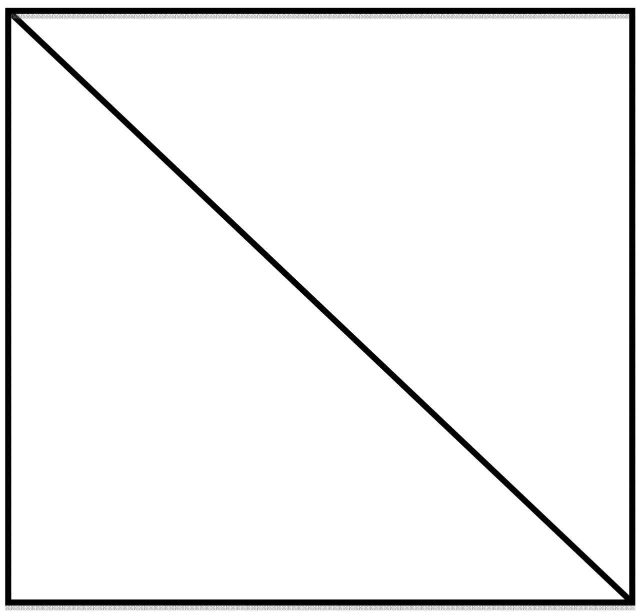


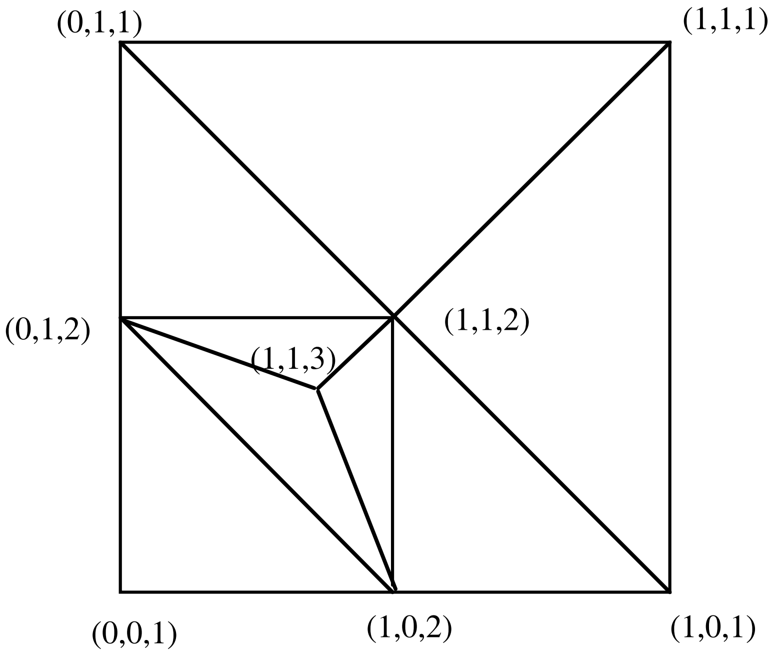


a second main result
(elimination of points of intederminacy in toric varieties)

THEOREM (de Concini-Procesi) *For any two regular triangulations* Δ *and* $\sum$ *on the same rational polyhedron, a sequence of blow-ups leads from* Δ *to a subdivision of* $\sum$

by successive blowing ups, we will be able to refine any rational triangulation





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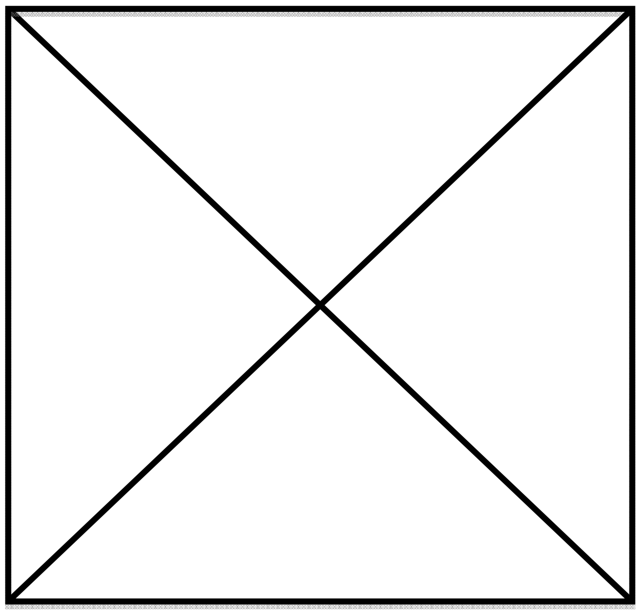

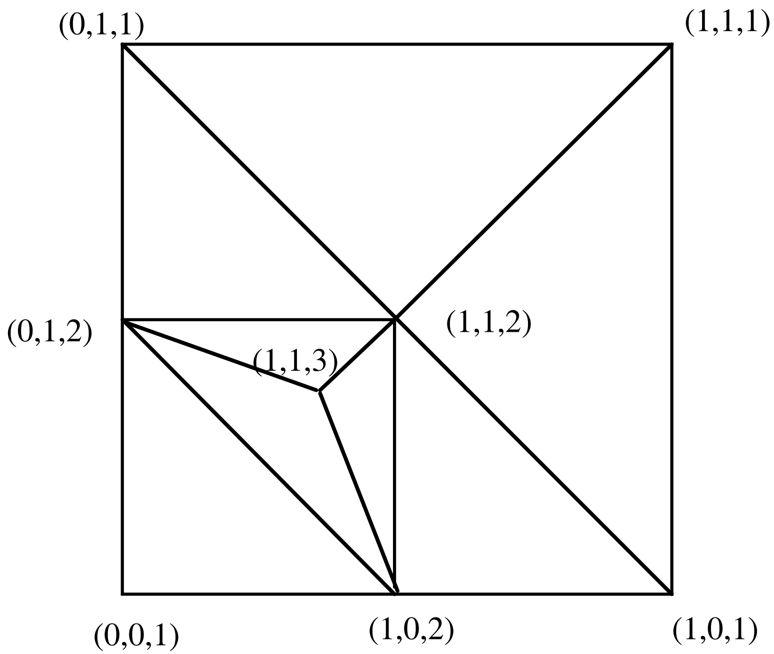

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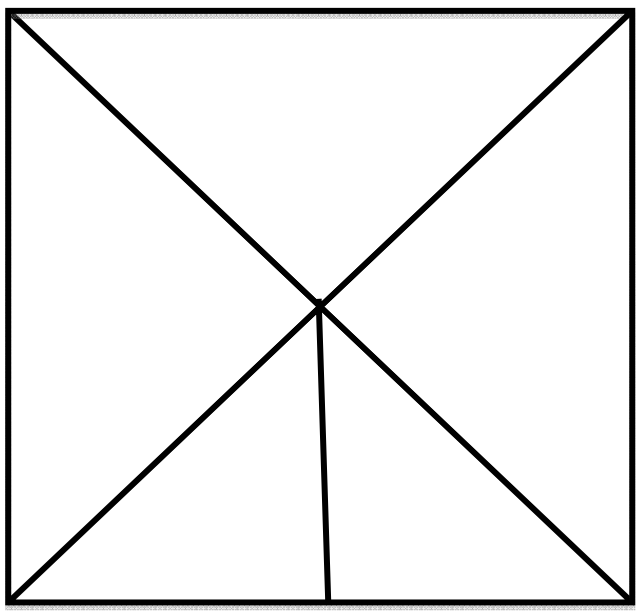

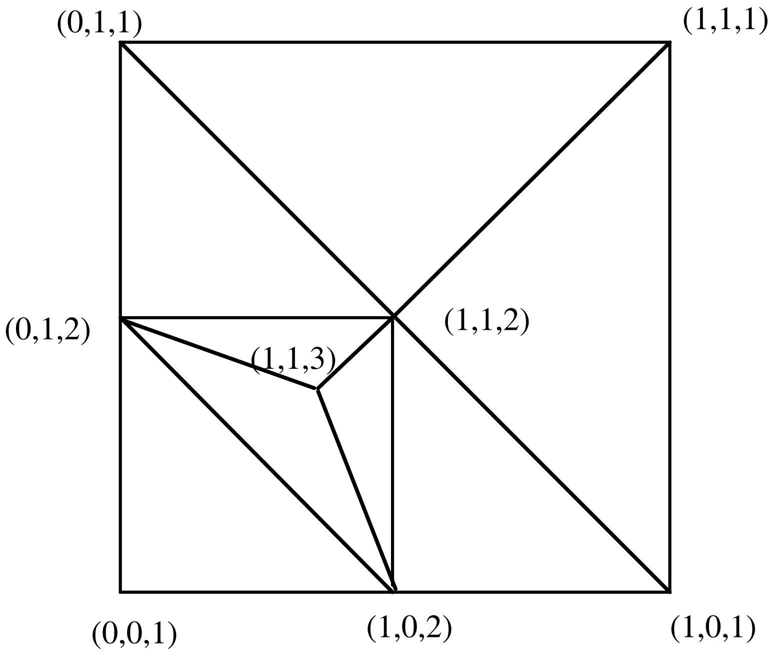

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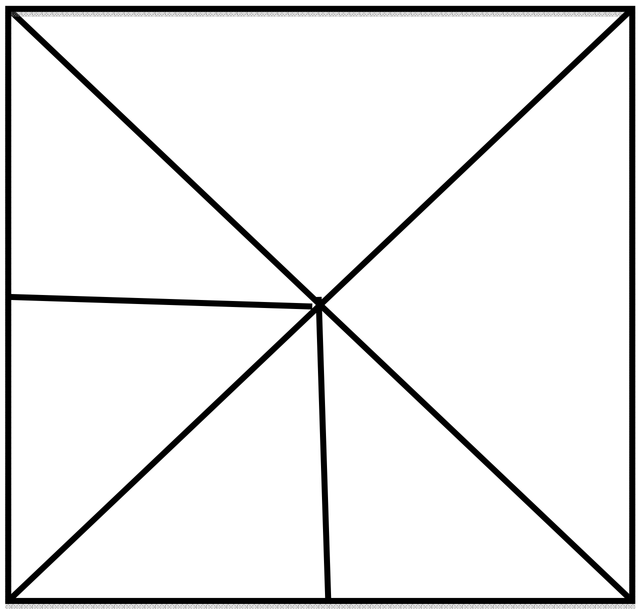

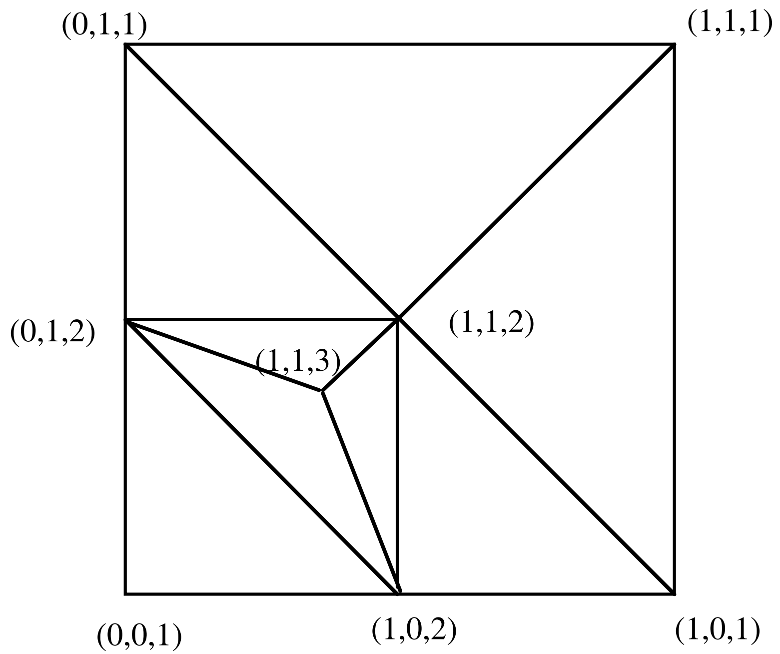

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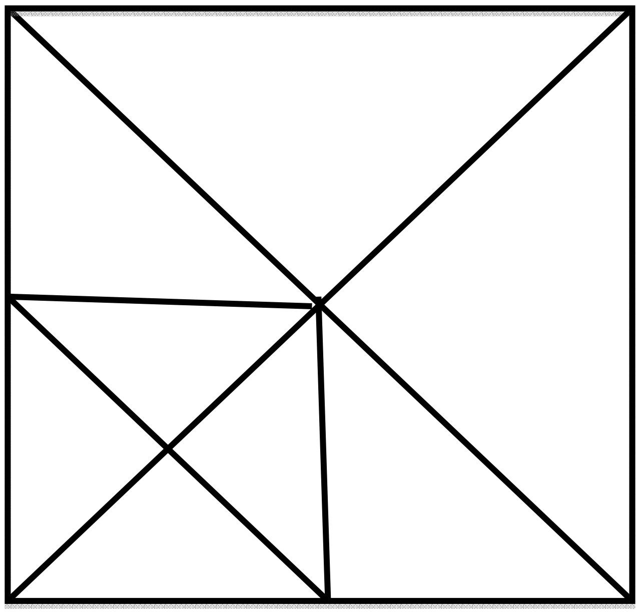

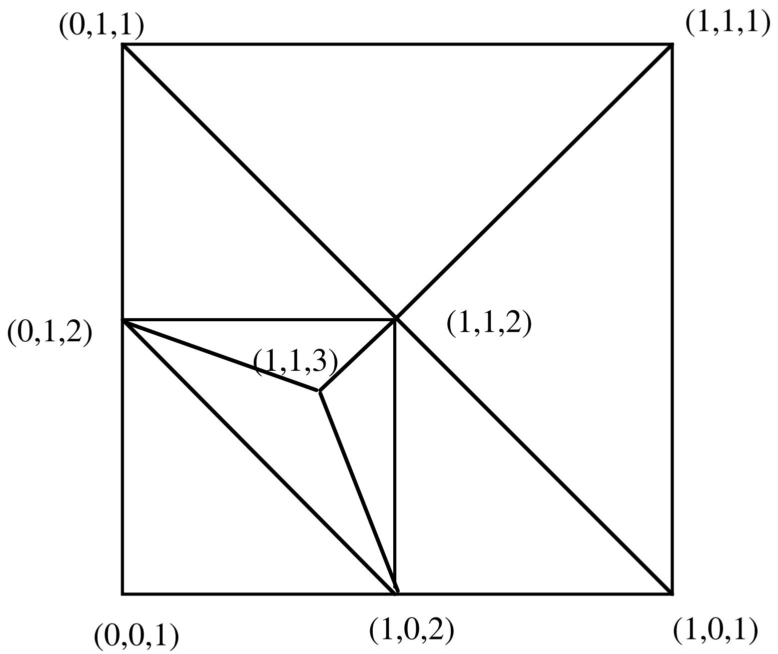

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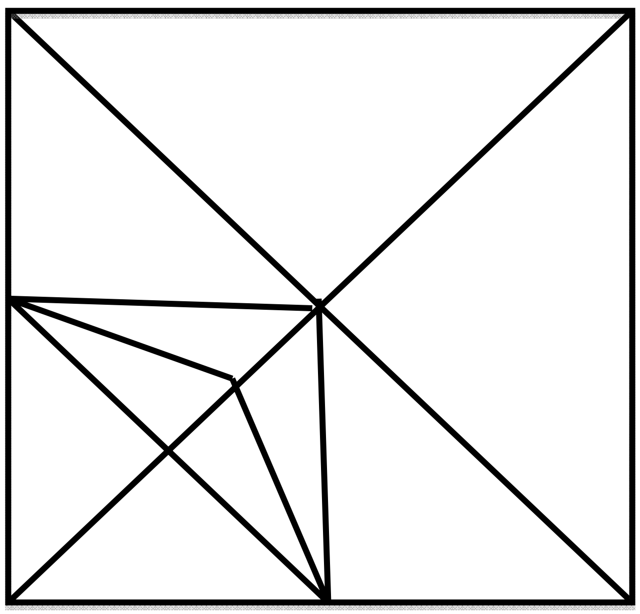
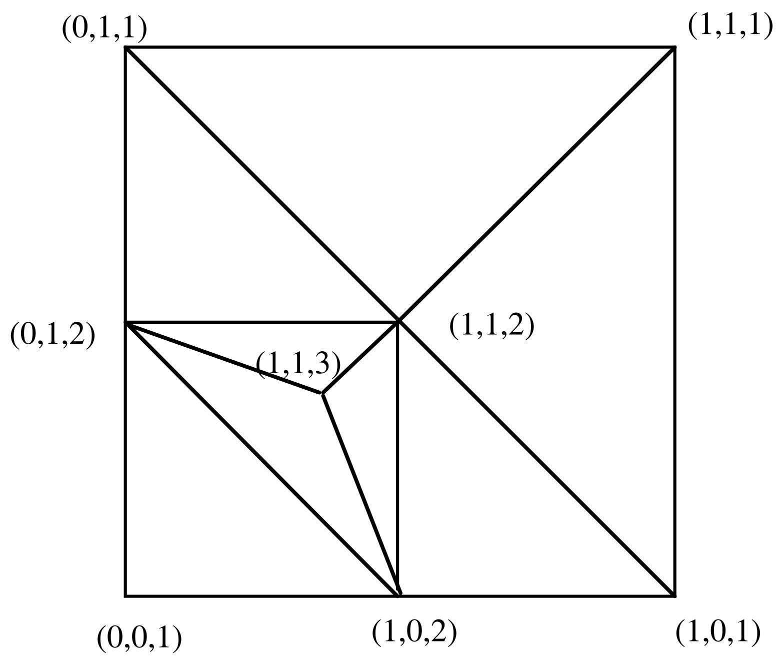
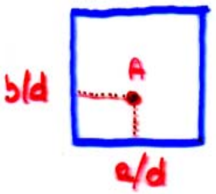


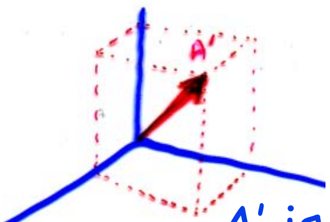


why toric varieties?

affine rational / homogeneous integer



A is a rational point of denominator d



A' is the integer vector d(A,1)

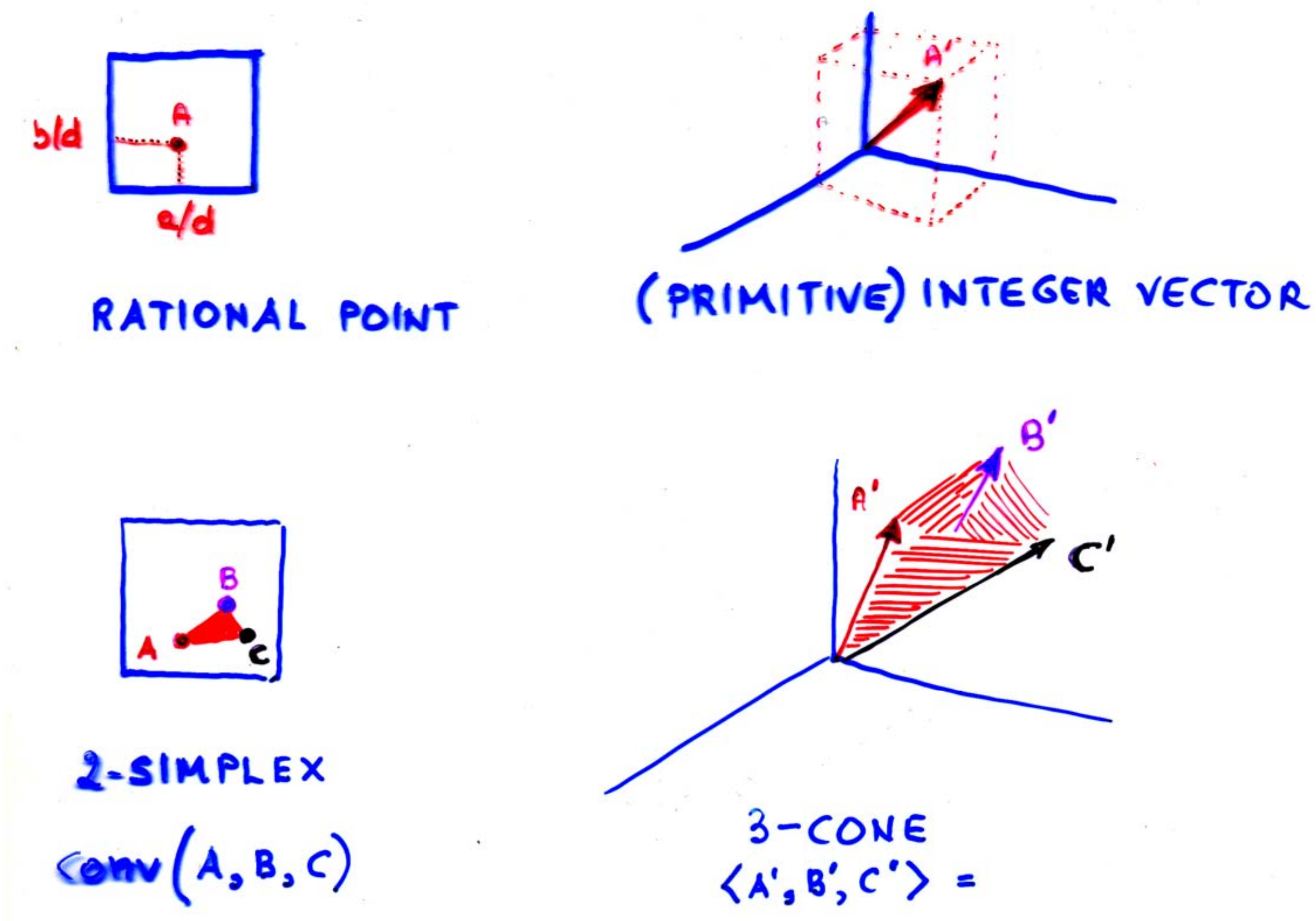
given a rational point
  A = $(x_1,\ldots,x_n)$ in $\mathbf{R}^n$

let **d** be the **denominator** of **A**

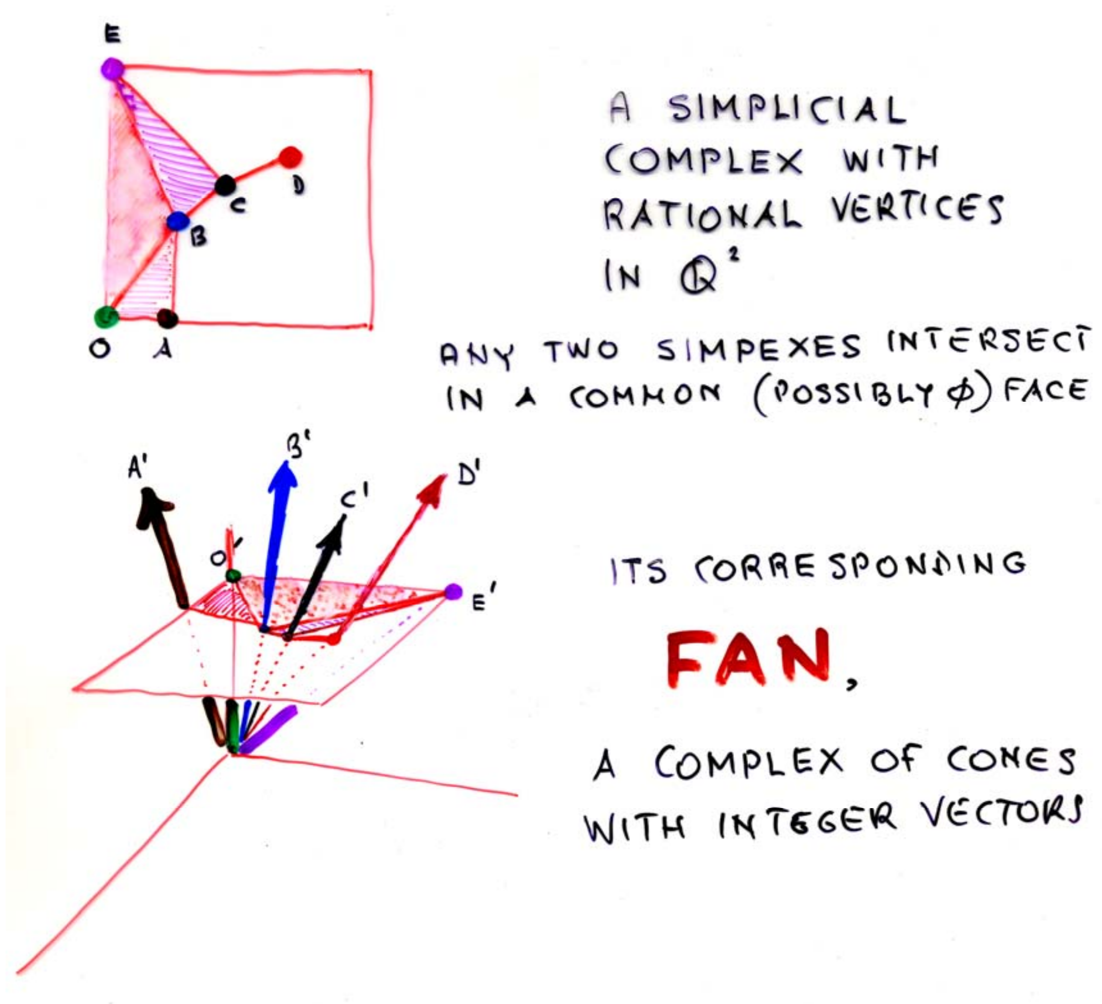
then the tuple $d(x_1,\ldots,x_n,1)$ is a vector **A'** in $\mathbf{Z}^{n+1}$

A' is called the homogeneous correspondent of **A**

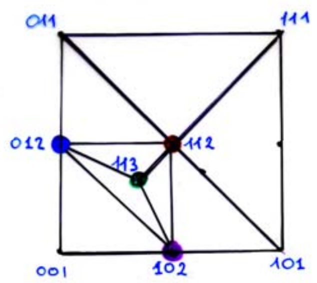
rational simplex <—-> integral cone



rational triangulation <—> fan

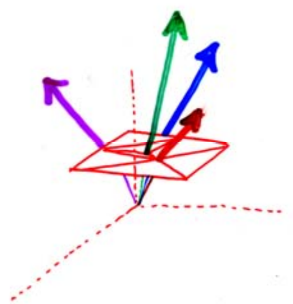


regular triangulation<—>regular fan



passing to homogeneous integer coordinates,

every regular (unimodular) triangulation determines

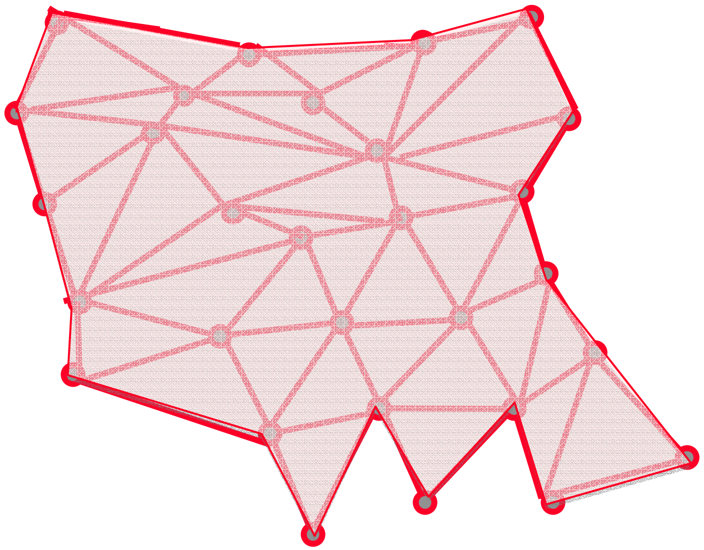


a regular (nonsingular, smooth) fan,

a standard tool in algebraic geometry

to code nonsingular toric varieties

properties of Vol(P) = ∑Vol(dΔ)

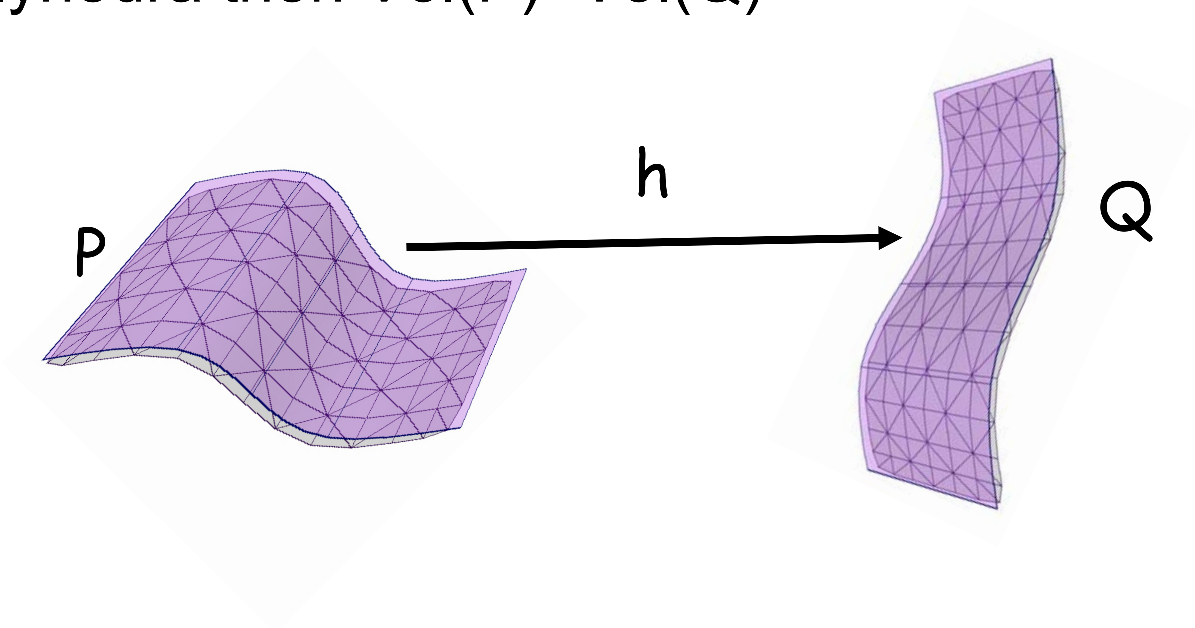


THEOREM ∑Vol(dΔ) does not depend on Δ. So the notation Vol(P) is unambiguous

This follows from the proof of Oda's conjecture, upon noting that Vol(P) is invariant under blow-ups

invariance under **Z**-homeomorphism

THEOREM If P and Q are **Z**-homeomorphic rational polyhedra then Vol(P)=Vol(Q)

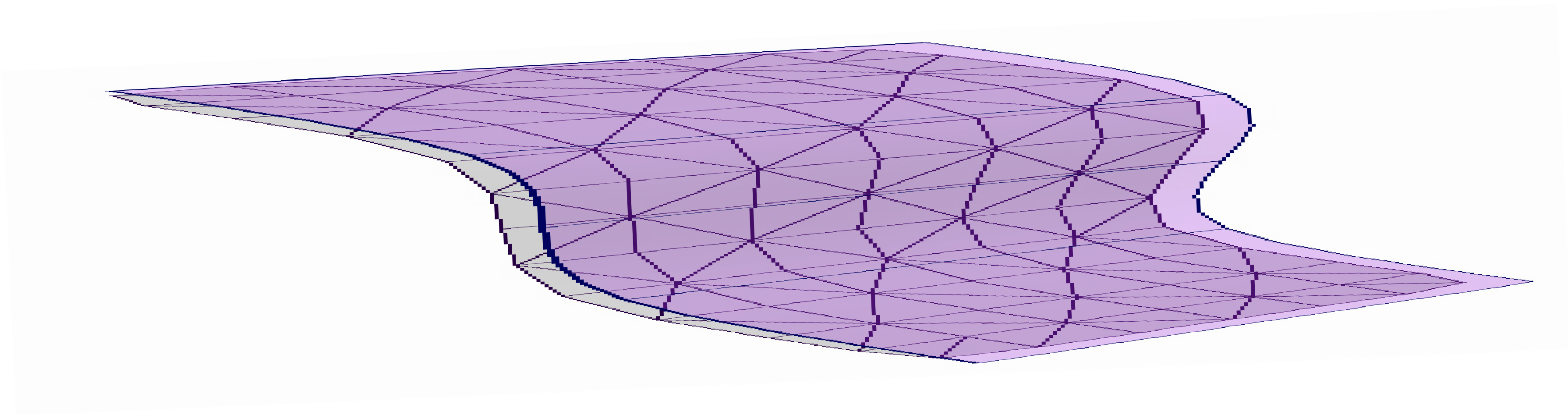


by the De Concini-Procesi theorem, given h we can always compute the volumes of P and Q with the help of a regular triangulation Δ of P such that h is linear over each simplex of Δ

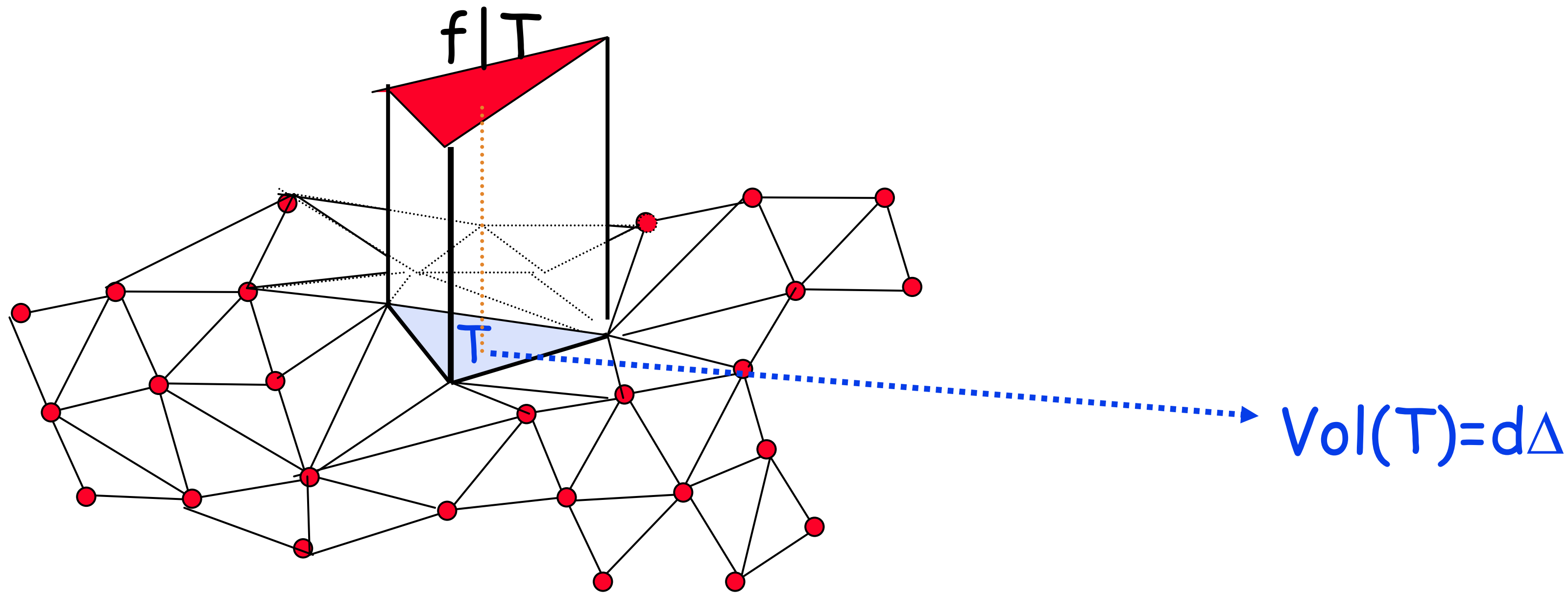
extends Lebesgue measure

THEOREM When P is full-dimensional,
$\sum \mathrm{Vol}(d\Delta)$ is the Lebesgue measure of P

THEOREM When P is Lebesgue-negligible (as a lower-dimensional polyhedron) still, $\sum \mathrm{Vol}(d\Delta)$ is nonzero

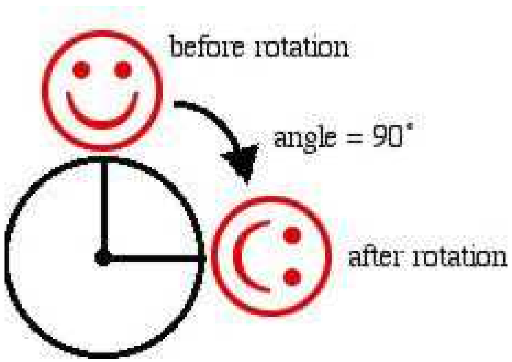


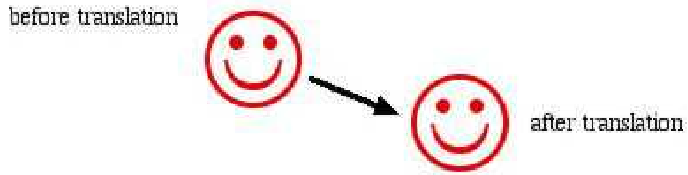
The integral of f over P is now defined in the natural way, as the volume underlying the graph of f



The regular triangulation Δ to compute the integral $\int_P f \, d\Delta$ is so chosen that f is linear on each simplex of Δ

We have thus attached to every rational polyhedron P a measure that is invariant under **Z**-homeomorphisms, coincides with Lebesgue measure if P is full-dimensional, but does not vanish if P is lower-dimensional





connections with logic (an introduction for non-logicians)

a main merit of classical logic

to give a rigorous meaning to the statement

conclusion p "follows" from premises $p_1, \ldots, p_n$

"consequence" becomes a mathematical notion

a main merit of L_∞

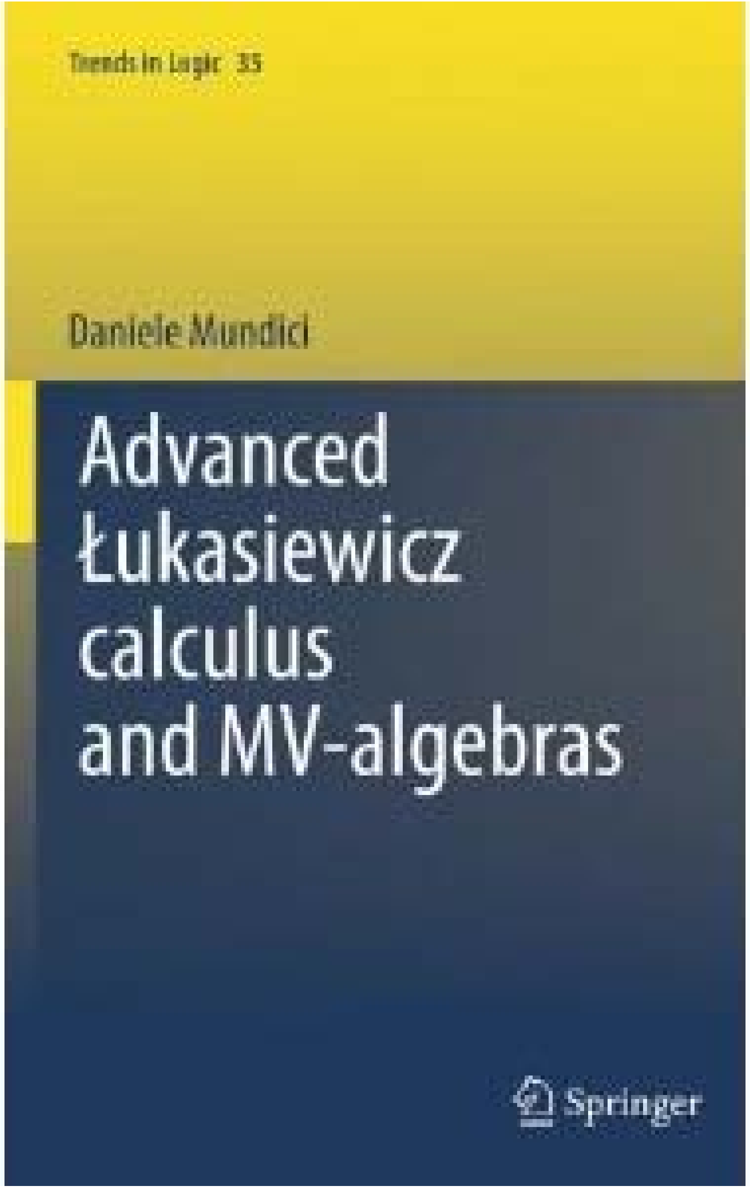
to give a rigorous meaning to the following statement:

p “stably” follows from premises $p_1,\dots,p_n$

in the sense that, even if we randomly delete a certain percentage of the formulas, formula **p** still follows (in the sense of the previous slide) from the remaining formulas **$p_1,\dots,p_n$**

L_∞ is a mathematically interesting logic for the treatment of partially unreliable information. L_∞ is the logic of the (Rènyi-Ulam) Twenty Questions game, where a certain number of answers may be distorted/wrong/mendacious

basic reference on Lukasiewicz logic L_∞

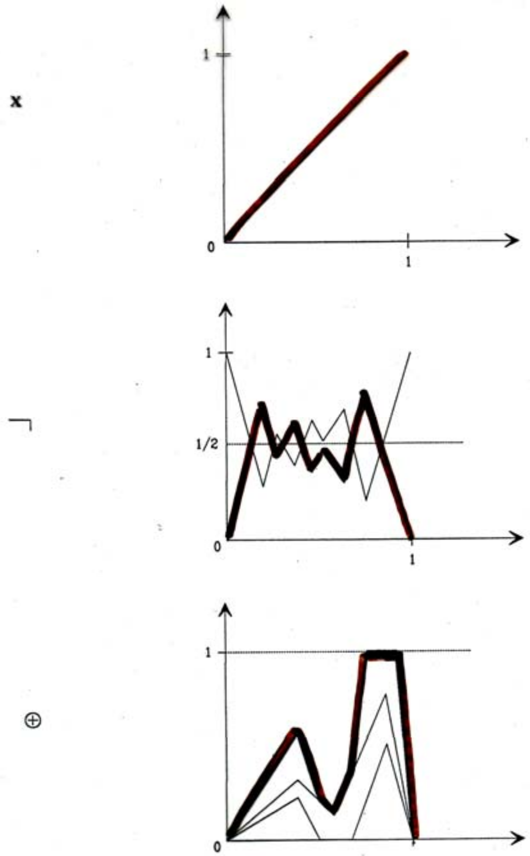


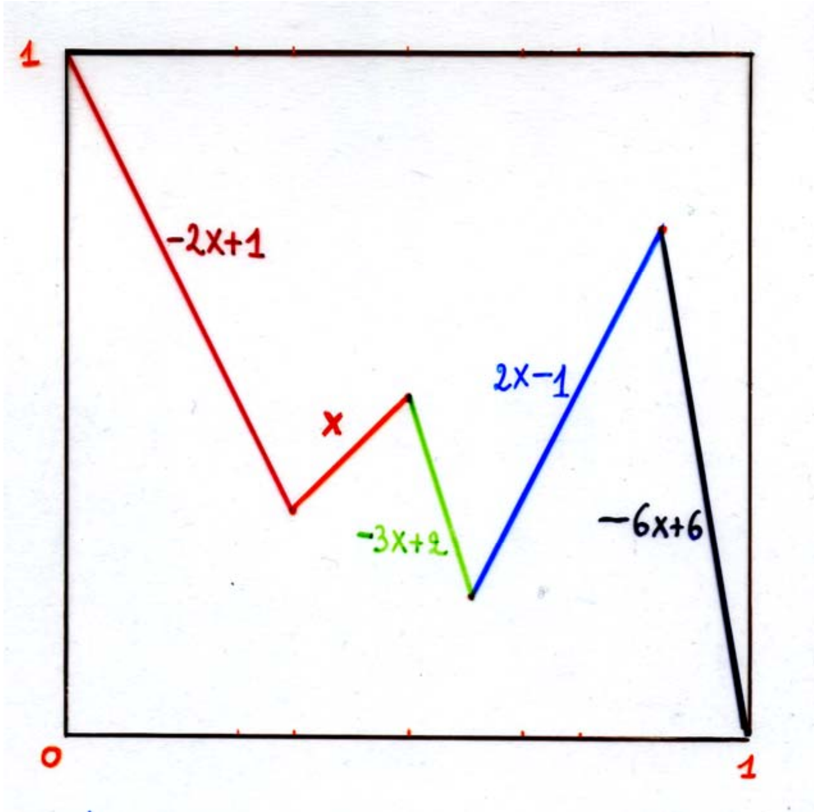
- any formula F in L_∞ describes the output of a continuous spectrum observable or event, just as a formula in classical boolean logic describes a yes-no event
- *Mod(F), the set of models of F*, is the most general rational polyhedron
- *Mod(T), for T a set of formulas*, is the most general compact Hausdorff space

the L_∞ language

- incorporates numbers and percentages in the language, without mentioning them
- we too, in everyday life, do not quantify our dubiousness degrees when reasoning informally
- rather we prefer to use adjectives or adverbs, like “uncertain” or “moderately unreliable”—and we are still able to make reasonable inferences
- only in classical logic and mathematical reasoning we assume 100% reliability

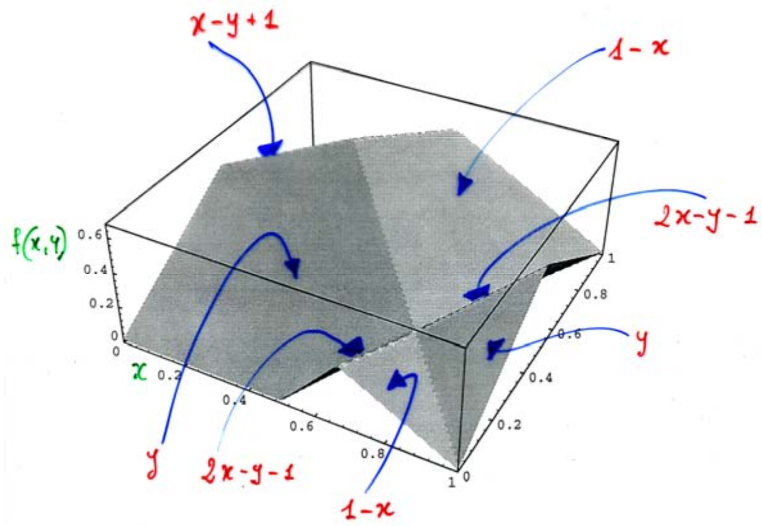
formulas in one variable



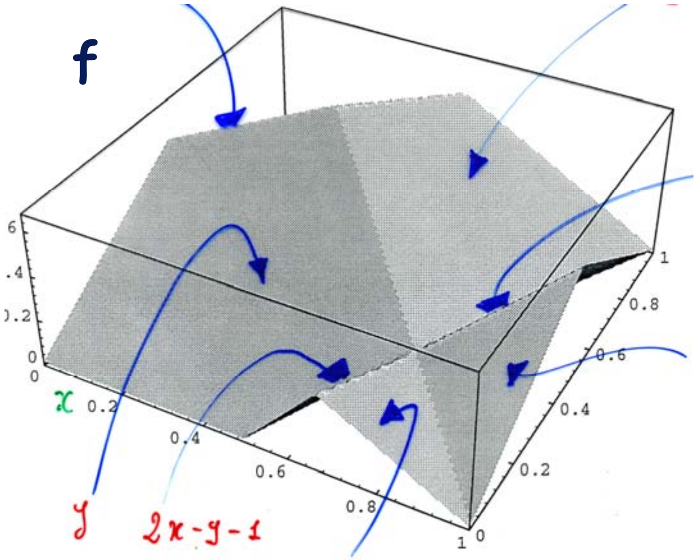


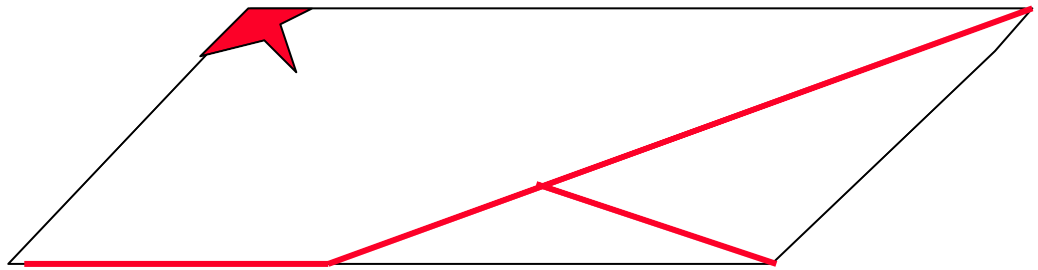
a formula f in two variables

- f is continuous
- f has finitely many linear pieces
- each piece of f has the form $a_1x_1+...+a_nx_n+b$
- where b and the a's are integers.
- Any function f with these properties is called a McNaughton function



the zeroset of f





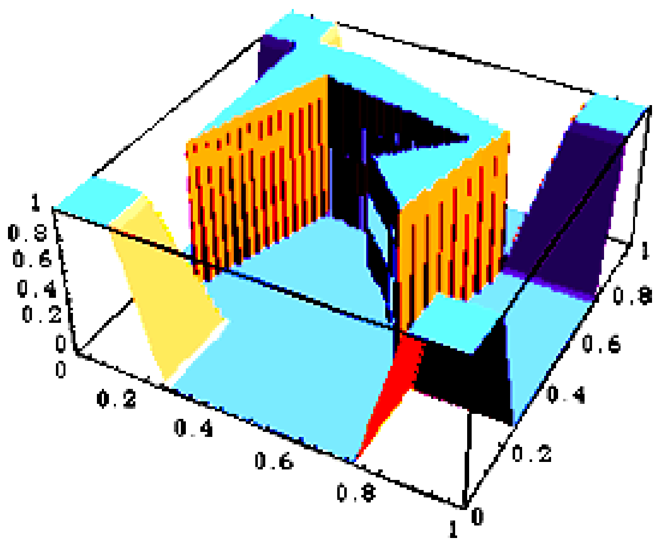
its zeroset
$Z(f) = f^{-1}(0)$

the domain of f can be decomposed into finitely many simplexes S_i in such a way that f is linear over each S_i

polyhedra as "affine varieties" of formulas

L_∞-formulas determine the most general possible rational polyhedron in $[0,1]^n$

rational polyhedra = "affine varieties" of L_∞-formulas



a formula F in L_∞ and its set of models Mod(F) = $F^{-1}(1)$ = set of truth-valuations that satisfy F

MV-algebras are the algebras of L_∞-formulas

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z$$

$$x \oplus y = y \oplus x$$

$$x \oplus 0 = x$$

$$\neg\neg x = x$$

$$x \oplus \neg 0 = \neg 0$$

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

these are the defining equations of MV-algebras

boolean algebras stand to classical logic as MV-algebras stand to L_∞

boolean algebras are obtained by adjoining the equation x+x=x

polyhedron=MV-presentation

- **COROLLARY** *Given a rational polyhedron P in the n-cube, let J(P) be the set of McNaughton functions of the free MV-algebra $FREE_n$ vanishing over P. Then J(P) is a principal ideal of the free algebra $FREE_n$*
- *Conversely, for every principal ideal J of $FREE_n$ let Z(J) be the intersection of the zerosets of all functions in J. Then Z(J) is a polyhedron in the n-cube, which coincides with the zeroset of any generator j of J*
- *The two maps P—>J(P) and J—>Z(J) are mutually inverse of each other*
- *these two maps induce a one-one correspondence between rational polyhedra and finitely presented MV-algebras*

closing a circle of ideas: invariant measures on polyhedra are in 1-1 correspondence with invariant probability measures on formulas

states in an MV-algebra A

- a **state** f of A is a normalized functional on A which is additive on incompatible elements of A
- THEOREM (Kroupa-Panti) The states of any MV-algebra A are in one-one correspondence with the regular Borel probability measures on the maximal space μ(A) of A
- thus the finitely additive algebraic notion of state corresponds to the usual notion of sigma-additive regular Borel probability

measures=states

- the ratio $\int_P f\, d\Delta / \int_P d\Delta$ is a *computable* rational number, once the function f is presented via a formula of Lukasiewicz logic
- this ratio does not depend on the regular triangulation Δ
- the map $f \longrightarrow \int_P f\, d\Delta / \int_P d\Delta$ is an **invariant** state of the finitely presented MV-algebra A(P) corresponding to P, called the **Lebesgue state** of A(P), and denoted $L_{A(P)}$
- a state f is **invariant** if f(a(x))=f(x) for every x in A(P) and automorphism a of A(P)

conditionals from the Lebesgue state

- let Q be a *variable* rational polyhedron in some cube $[0,1]^n$. This Q is the model-set of a formula G in Lukasiewicz logic.
- given any other formula F with its McNaughton function f_F, the integral of f_F over Q, divided by Vol(Q) is a *conditional probability* **P**(F|*G*) *of F given G*
- **P**(F|*G*) has various properties: *rationality, computability, invariance, substitutability*: **P**(B|C) = **P**(X|(C&X⇔B))
- and also satisfies *Rényi's "law of compound probabilities"*, which for yes-no events reads:
- **P**(A&B|C) = **P**(A|B&C) · **P**(B|C)

this talk was only aimed at showing that the notion of **Z**-homeomorphism is dual to the notion of MV-algebraic isomorphism, and thus comes from Lukasiewicz logic

Z-homeomorphism is at the very beginning of an extensive and deep theory, involving fans, ordered groups, abstract simplicial complexes, probability theory and C*-algebras

MV-algebras and their states inside mathematics

CHANG MV-ALGEBRAS	06D35
=VIA Γ FUNCTOR	
ABELIAN ℓ-GROUPS WITH UNIT	06F20
=VIA K_0	
AF C*-ALGEBRAS WITH LATTICE-ORDERED MURRAY VON NEUMANN ORDER	46L80

PIECEWISE LINEAR FUNCTIONS WITH INTEGER COEFFICIENTS (FREE MV-ALG.)	57Q05
TORIC DESINGULARIZATION	14M25

- countable MV-algebras correspond to those AF C*-algebras whose Murray-von Neumann order of projections is a lattice.
- Invariant states of MV-algebras yield invariant states on their corresponding AF C*-algebras

Thank you

M. Busaniche, D.M., Geometry of Robinson consistency in Łukasiewicz logic, *Annals of Pure and Applied Logic, 147 (2007) 1-22.*

M.Busaniche, L.Cabrer, D.M., Confluence and combinatorics in finitely generated unital lattice-ordered abelian groups, *Forum Mathematicum, doi: 10.1515/FORM.2011.059*

D.M., Finite axiomatizability in Lukasiewicz logic, *Annals of Pure and Applied Logic, doi: 10.1016/j.apal.2011.06.026*

D.M. Advanced Łukasiewicz calculus and MV-algebras, Trends in Logic, Vol. 35 Springer, New York, (2011).