Wavelets y regularidad Besov en temperaturas

Seminario “Carlos Segovia Fernández”

10 de junio de 2011

Wavelets

Wavelets

- $\mathcal{D}_j$, $j \in \mathbb{Z}$, cubos diádicos en $I\!R^d$ de medida 2^{-jd}

Wavelets

- $\mathcal{D}_j$, $j \in \mathbb{Z}$, cubos diádicos en $I\!R^d$ de medida 2^{-jd}
- $\mathcal{D} = \cup_{j \in \mathbb{Z}} \mathcal{D}_j$ familia de todos los cubos diádicos en $I\!R^d$ y $\mathcal{D}^+$ los de medida ≤ 1

Wavelets

- $\mathcal{D}_j$, $j \in \mathbb{Z}$, cubos diádicos en $I\!R^d$ de medida 2^{-jd}
- $\mathcal{D} = \cup_{j\in\mathbb{Z}}\mathcal{D}_j$ familia de todos los cubos diádicos en $I\!R^d$ y $\mathcal{D}^+$ los de medida ≤ 1
- Para n entero positivo existe $\Psi \subset \mathscr{C}_0^n(I\!R^d)$, $\#\Psi = 2^d - 1$, con n-momentos nulos, $\phi \in \mathscr{C}_0^n(I\!R^d)$, $\operatorname{sop}\phi \subset Q$, tal que

$$\{\psi_I : \psi \in \Psi, I \in \mathcal{D}\} \quad \text{b.o.n. de } L^2(I\!R^d)$$

- $\psi_I(x) = 2^{\frac{jd}{2}}\psi(2^j x - k)$, $I = I_k^j$, $k = (k_1, \dots, k_d)$, $\operatorname{sop}\psi_I \subset Q(I)$

Wavelets

Wavelets

Para $\psi \in \Psi$, y $I \in \mathcal{D}$

$$\psi_{I,p} = |I|^{\frac{1}{2}-\frac{1}{p}}\,\psi_I$$

Notar $\psi_I = \psi_{I,2}$ y $\|\psi_{I,p}\|_{L_p(\mathbb{R}^d)} = \|\psi\|_{L_p(\mathbb{R}^d)}$ para todo $I \in \mathcal{D}$.

Wavelets

Para $\psi \in \Psi$, y $I \in \mathcal{D}$

$$\psi_{I,p} = |I|^{\frac{1}{2}-\frac{1}{p}} \psi_I$$

Notar $\psi_I = \psi_{I,2}$ y $\|\psi_{I,p}\|_{L_p(\mathbb{R}^d)} = \|\psi\|_{L_p(\mathbb{R}^d)}$ para todo $I \in \mathcal{D}$.

$$f = P_0 f + \sum_{I \in \mathcal{D}^+} \sum_{\psi \in \Psi} \langle f, \psi_{I,p'} \rangle \psi_{I,p}$$

P_0 es la proyección ortogonal sobre $S_0 = \overline{\text{span}\{\phi_I : I \in \mathcal{D}_0\}}$, y $\frac{1}{p} + \frac{1}{p'} = 1$.

Caracterización por medio de wavelets

Caracterización por medio de wavelets

Sean d, p, λ, α and τ como antes. Supongamos que $\Psi \subset \mathscr{C}^n(\mathbb{R}^d)$ para $n > \lambda + d$.

(A) $f \in B_p^\lambda(\mathbb{R}^d)$ si y sólo si

$$\|P_0 f\|_{L_p(\mathbb{R}^d)} + \left(\sum_{I \in \mathcal{D}^+} \sum_{\psi \in \Psi} |I|^{-\frac{\lambda p}{d}} \, |\langle f, \psi_{I,p'} \rangle|^p \right)^{\frac{1}{p}} < \infty$$

Caracterización por medio de wavelets

Sean d, p, λ, α and τ como antes. Supongamos que $\Psi \subset \mathscr{C}^n(\mathbb{R}^d)$ para $n > \lambda + d$.

(A) $f \in B_p^\lambda(\mathbb{R}^d)$ si y sólo si

$$\|P_0 f\|_{L_p(\mathbb{R}^d)} + \left(\sum_{I \in \mathcal{D}^+} \sum_{\psi \in \Psi} |I|^{-\frac{\lambda p}{d}} \, |\langle f, \psi_{I,p'} \rangle|^p \right)^{\frac{1}{p}} < \infty$$

(B) $f \in B_\tau^\alpha(\mathbb{R}^d)$ si y sólo si

$$\|P_0 f\|_{L_\tau(\mathbb{R}^d)} + \left(\sum_{I \in \mathcal{D}^+} \sum_{\psi \in \Psi} |\langle f, \psi_{I,p'} \rangle|^\tau \right)^{\frac{1}{\tau}} < \infty$$

Dahlke-DeVore

Si D es un dominio Lipschitz en $\mathbb{R}^d$, $1 < p < \infty$, $\lambda > 0$, $0 < \alpha < \frac{\lambda d}{d-1}$ y $\frac{1}{\tau} = \frac{1}{p} + \frac{\alpha}{d}$, entonces

$$\mathscr{H}(D) \cap B_p^\lambda(D) \subset B_\tau^\alpha(D)$$

donde $\mathscr{H}(D) = \{u : \Delta u = 0 \text{ en } D\}$

Temperaturas

Temperaturas

- $\Omega = D \times (0, T)$, $T > 0$, D Lipschitz acotado de $\mathbb{R}^d$

Temperaturas

- $\Omega = D \times (0, T)$, $T > 0$, D Lipschitz acotado de $\mathbb{R}^d$
- $\Theta(\Omega) = \{u : \frac{\partial u}{\partial t} = \Delta u \text{ en } \Omega\}$

Temperaturas

- $\Omega = D \times (0, T)$, $T > 0$, D Lipschitz acotado de $\mathbb{R}^d$
- $\Theta(\Omega) = \{u : \frac{\partial u}{\partial t} = \Delta u \text{ en } \Omega\}$
- Para $0 < \gamma < 1$, $1 < r < \infty$ definimos

$$\mathbb{B}_r^\gamma(\Omega) = \left(L_r(\Omega), W_r^{2,1}(\Omega)\right)_{\frac{\gamma}{2}, r}$$

Temperaturas

- $\Omega = D \times (0,T)$, $T > 0$, D Lipschitz acotado de $\mathbb{R}^d$
- $\Theta(\Omega) = \{u : \frac{\partial u}{\partial t} = \Delta u \text{ en } \Omega\}$
- Para $0 < \gamma < 1$, $1 < r < \infty$ definimos

$$\mathbb{B}_r^\gamma(\Omega) = \left(L_r(\Omega), W_r^{2,1}(\Omega)\right)_{\frac{\gamma}{2},r}$$

$$\|v\|_{W_r^{2,1}(\Omega)} = \|v\|_{L_r(\Omega)} + \sum_{i=1}^{d} \left\|\frac{\partial v}{\partial x_i}\right\|_{L_r(\Omega)} + \sum_{i=1}^{d}\sum_{j=1}^{d} \left\|\frac{\partial^2 v}{\partial x_i \partial x_j}\right\|_{L_r(\Omega)} + \left\|\frac{\partial v}{\partial t}\right\|_{L_r(\Omega)}$$

Temperaturas

- $\Omega = D \times (0,T)$, $T > 0$, D Lipschitz acotado de $\mathbb{R}^d$
- $\Theta(\Omega) = \{u : \frac{\partial u}{\partial t} = \Delta u \text{ en } \Omega\}$
- Para $0 < \gamma < 1$, $1 < r < \infty$ definimos

$$\mathbb{B}_r^\gamma(\Omega) = \left(L_r(\Omega), W_r^{2,1}(\Omega)\right)_{\frac{\gamma}{2},r}$$

$$\|v\|_{W_r^{2,1}(\Omega)} = \|v\|_{L_r(\Omega)} + \sum_{i=1}^{d} \left\|\frac{\partial v}{\partial x_i}\right\|_{L_r(\Omega)} + \sum_{i=1}^{d}\sum_{j=1}^{d} \left\|\frac{\partial^2 v}{\partial x_i \partial x_j}\right\|_{L_r(\Omega)} + \left\|\frac{\partial v}{\partial t}\right\|_{L_r(\Omega)}$$

- $\tilde{\delta}(x) = \text{ínf}\{|x-y| : y \in \partial D\}$
- $\rho((x,t);(y,s)) = \text{máx}\{|x-y|, \sqrt{|t-s|}\}$ en $\mathbb{R}^{d+1}$

Temperaturas

- $\Omega = D \times (0,T)$, $T > 0$, D Lipschitz acotado de $\mathbb{R}^d$
- $\Theta(\Omega) = \{u : \frac{\partial u}{\partial t} = \Delta u \text{ en } \Omega\}$
- Para $0 < \gamma < 1$, $1 < r < \infty$ definimos

$$\mathbb{B}_r^\gamma(\Omega) = \left(L_r(\Omega), W_r^{2,1}(\Omega)\right)_{\frac{\gamma}{2},r}$$

$$\|v\|_{W_r^{2,1}(\Omega)} = \|v\|_{L_r(\Omega)} + \sum_{i=1}^{d} \left\| \frac{\partial v}{\partial x_i} \right\|_{L_r(\Omega)} + \sum_{i=1}^{d} \sum_{j=1}^{d} \left\| \frac{\partial^2 v}{\partial x_i \partial x_j} \right\|_{L_r(\Omega)} + \left\| \frac{\partial v}{\partial t} \right\|_{L_r(\Omega)}$$

- $\tilde{\delta}(x) = \text{ínf}\{|x-y| : y \in \partial D\}$
- $\rho((x,t);(y,s)) = \text{máx}\{|x-y|, \sqrt{|t-s|}\}$ en $\mathbb{R}^{d+1}$
- $\partial_{\text{par}}\Omega = (D \times \{0\}) \cup (\partial D \times [0,T))$
- $\delta(x,t) = \text{ínf}\{\rho((x,t);(y,s)) : (y,s) \in \partial_{\text{par}}\Omega\}$

(AGI 2008)

Sea $0 < \lambda < \ell < \lambda + d$, $1 < p < \infty$. *Para alguna constante* C *y para toda* $u \in \Theta(\Omega)$,

$$\left\| \delta^{\ell-\lambda} \left| \nabla^\ell u \right| \right\|_{L_p(\Omega)} \leq C \left\| u \right\|_{L_p((0,T);B_p^\lambda(D))}$$

(AGI 2008)

Sea $0 < \lambda < \ell < \lambda + d$, $1 < p < \infty$. *Para alguna constante* C *y para toda* $u \in \Theta(\Omega)$,

$$\left\| \delta^{\ell-\lambda} \left| \nabla^{\ell} u \right| \right\|_{L_p(\Omega)} \leq C \left\| u \right\|_{L_p((0,T);B_p^{\lambda}(D))}$$

$$\|u\|_{L_p((0,T);B_p^{\lambda}(D))} = \left(\int_0^T \|u(\cdot,t)\|_{B_p^{\lambda}(D)}^p \, dt \right)^{\frac{1}{p}}$$

(AGI 2008)

Sea $0 < \lambda < \ell < \lambda + d$, $1 < p < \infty$. *Para alguna constante* C *y para toda* $u \in \Theta(\Omega)$,

$$\left\| \delta^{\ell-\lambda} \left| \nabla^{\ell} u \right| \right\|_{L_p(\Omega)} \leq C \, \| u \|_{L_p((0,T);B_p^{\lambda}(D))}$$

$$\| u \|_{L_p((0,T);B_p^{\lambda}(D))} = \left(\int_0^T \| u(\cdot,t) \|_{B_p^{\lambda}(D)}^p \, dt \right)^{\frac{1}{p}}$$

(AGI 2010)

Para $\gamma > 0$, $1 < q < \infty$ *y* $0 < \varepsilon < \gamma$

$$\Theta(\Omega) \cap L_q((0,T); B_q^{\gamma}(D)) \subset \mathbb{B}_q^{\gamma-\varepsilon}(\Omega)$$

Mejora de la regularidad en temperaturas

Mejora de la regularidad en temperaturas

Sean $1 < p < \infty$, $\lambda > 0$, $\ell = [\lambda + d]$, $0 < \alpha < \text{mín}\{\ell, \frac{\lambda d}{d-1}\}$ *y* $\frac{1}{\tau} = \frac{1}{p} + \frac{\alpha}{d}$. *Entonces*

$$\Theta(\Omega) \cap L_p((0,T); B_p^\lambda(D)) \subset L_\tau((0,T); B_\tau^\alpha(D))$$

Mejora de la regularidad en temperaturas

Sean $1 < p < \infty$, $\lambda > 0$, $\ell = [\lambda + d]$, $0 < \alpha < \min\{\ell, \frac{\lambda d}{d-1}\}$ *y* $\frac{1}{\tau} = \frac{1}{p} + \frac{\alpha}{d}$. *Entonces*

$$\Theta(\Omega) \cap L_p((0,T); B_p^\lambda(D)) \subset L_\tau((0,T); B_\tau^\alpha(D))$$

Sean $1 < p < \infty$, $\lambda > 0$, $0 < \alpha < \min\{d\big(1 - \frac{1}{p}\big), \frac{\lambda d}{d-1}\}$ *y* $\frac{1}{\tau} = \frac{1}{p} + \frac{\alpha}{d}$. *Entonces*

$$\Theta(\Omega) \cap \mathbb{B}_p^\lambda(\Omega) \subset \bigcap_{\alpha > \varepsilon > 0} \mathbb{B}_\tau^{\alpha-\varepsilon}(\Omega)$$

Partición del conjunto de índices wavelet

- $\delta(x,t) = \min\{\tilde{\delta}(x), \sqrt{t}\}$

Partición del conjunto de índices wavelet

- $\delta(x,t) = \text{mín}\{\tilde{\delta}(x), \sqrt{t}\}$
- $\Omega^1 = \{(x,t) \in \Omega : \delta(x,t) = \tilde{\delta}(x)\}$,
  $\Omega^2 = \{(x,t) \in \Omega : \delta(x,t) = \sqrt{t}\}^o$

Partición del conjunto de índices wavelet

- $\delta(x,t) = \text{mín}\{\tilde{\delta}(x), \sqrt{t}\}$
- $\Omega^1 = \{(x,t) \in \Omega : \delta(x,t) = \tilde{\delta}(x)\}$,
  $\Omega^2 = \{(x,t) \in \Omega : \delta(x,t) = \sqrt{t}\}^o$
- $\Gamma_j = \{I \in \mathcal{D}_j : Q(I) \cap \overline{D} \neq \varnothing\}$, $j \geq 0$

Partición del conjunto de índices wavelet

- $\delta(x,t) = \text{mín}\{\tilde{\delta}(x), \sqrt{t}\}$
- $\Omega^1 = \{(x,t) \in \Omega : \delta(x,t) = \tilde{\delta}(x)\}$,
  $\Omega^2 = \{(x,t) \in \Omega : \delta(x,t) = \sqrt{t}\}^o$
- $\Gamma_j = \{I \in \mathcal{D}_j : Q(I) \cap \overline{D} \neq \varnothing\}$, $j \geq 0$
- Cubos frontera: $\Gamma_{j,0} = \{I \in \mathcal{D}_j : Q(I) \cap \partial D \neq \varnothing\}$

Partición del conjunto de índices wavelet

- $\delta(x,t) = \text{mín}\{\tilde{\delta}(x), \sqrt{t}\}$
- $\Omega^1 = \{(x,t) \in \Omega : \delta(x,t) = \tilde{\delta}(x)\}$,
  $\Omega^2 = \{(x,t) \in \Omega : \delta(x,t) = \sqrt{t}\}^o$
- $\Gamma_j = \{I \in \mathcal{D}_j : Q(I) \cap \overline{D} \neq \varnothing\}$, $j \geq 0$
- Cubos frontera: $\Gamma_{j,0} = \{I \in \mathcal{D}_j : Q(I) \cap \partial D \neq \varnothing\}$
- Cubos interiores $\overset{o}{\Gamma}_j = \Gamma_j \setminus \Gamma_{j,0}$: $0 < t < T$, $I(t) = I \times \{t\}$

$$\overset{o}{\Gamma}_j{}^1(t) = \{I \in \overset{o}{\Gamma}_j : I(t) \subset \Omega^1\} \quad \text{y} \quad \overset{o}{\Gamma}_j{}^2(t) = \overset{o}{\Gamma}_j \setminus \overset{o}{\Gamma}_j{}^1(t)$$

Partición del conjunto de índices wavelet

- $\delta(x,t) = \text{mín}\{\tilde{\delta}(x), \sqrt{t}\}$
- $\Omega^1 = \{(x,t) \in \Omega : \delta(x,t) = \tilde{\delta}(x)\}$,
  $\Omega^2 = \{(x,t) \in \Omega : \delta(x,t) = \sqrt{t}\}^o$
- $\Gamma_j = \{I \in \mathcal{D}_j : Q(I) \cap \overline{D} \neq \varnothing\}$, $j \geq 0$
- Cubos frontera: $\Gamma_{j,0} = \{I \in \mathcal{D}_j : Q(I) \cap \partial D \neq \varnothing\}$
- Cubos interiores $\overset{o}{\Gamma}_j = \Gamma_j \setminus \Gamma_{j,0}$: $0 < t < T$, $I(t) = I \times \{t\}$

$$\overset{o}{\Gamma}_j^{\,1}(t) = \{I \in \overset{o}{\Gamma}_j : I(t) \subset \Omega^1\} \quad \text{y} \quad \overset{o}{\Gamma}_j^{\,2}(t) = \overset{o}{\Gamma}_j \setminus \overset{o}{\Gamma}_j^{\,1}(t)$$

- $\Gamma_{j,k} = \{I \in \Gamma_j : k2^{-j} \leq \tilde{\delta}_{Q(I)} < (k+1)2^{-j}\}$, $k, j \geq 0$

Partición del conjunto de índices wavelet

- $\delta(x,t) = \text{mín}\{\tilde{\delta}(x), \sqrt{t}\}$
- $\Omega^1 = \{(x,t) \in \Omega : \delta(x,t) = \tilde{\delta}(x)\}$,
  $\Omega^2 = \{(x,t) \in \Omega : \delta(x,t) = \sqrt{t}\}^o$
- $\Gamma_j = \{I \in \mathcal{D}_j : Q(I) \cap \overline{D} \neq \varnothing\}$, $j \geq 0$
- Cubos frontera: $\Gamma_{j,0} = \{I \in \mathcal{D}_j : Q(I) \cap \partial D \neq \varnothing\}$
- Cubos interiores $\overset{o}{\Gamma}_j = \Gamma_j \setminus \Gamma_{j,0}$: $0 < t < T$, $I(t) = I \times \{t\}$

$$\overset{o}{\Gamma}_j^{\,1}(t) = \{I \in \overset{o}{\Gamma}_j : I(t) \subset \Omega^1\} \quad \text{y} \quad \overset{o}{\Gamma}_j^{\,2}(t) = \overset{o}{\Gamma}_j \setminus \overset{o}{\Gamma}_j^{\,1}(t)$$

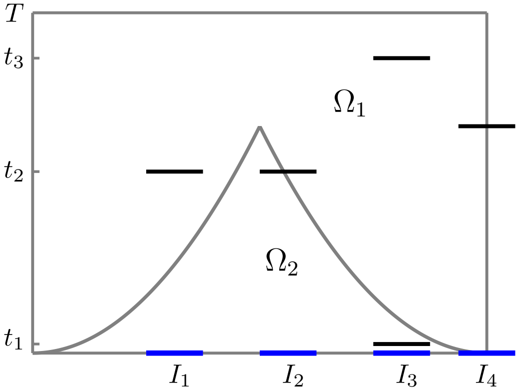
- $\Gamma_{j,k} = \{I \in \Gamma_j : k2^{-j} \leq \tilde{\delta}_{Q(I)} < (k+1)2^{-j}\}$, $k, j \geq 0$
- $\Gamma_j = \cup_{k \geq 0} \Gamma_{j,k}$

Partición del conjunto de índices wavelet

- $\delta(x,t) = \text{mín}\{\tilde{\delta}(x), \sqrt{t}\}$
- $\Omega^1 = \{(x,t) \in \Omega : \delta(x,t) = \tilde{\delta}(x)\}$,
  $\Omega^2 = \{(x,t) \in \Omega : \delta(x,t) = \sqrt{t}\}^o$
- $\Gamma_j = \{I \in \mathcal{D}_j : Q(I) \cap \overline{D} \neq \varnothing\}$, $j \geq 0$
- Cubos frontera: $\Gamma_{j,0} = \{I \in \mathcal{D}_j : Q(I) \cap \partial D \neq \varnothing\}$
- Cubos interiores $\overset{o}{\Gamma}_j = \Gamma_j \setminus \Gamma_{j,0}$: $0 < t < T$, $I(t) = I \times \{t\}$

$$\overset{o}{\Gamma}{}_j^{1}(t) = \{I \in \overset{o}{\Gamma}_j : I(t) \subset \Omega^1\} \quad \text{y} \quad \overset{o}{\Gamma}{}_j^{2}(t) = \overset{o}{\Gamma}_j \setminus \overset{o}{\Gamma}{}_j^{1}(t)$$

- $\Gamma_{j,k} = \{I \in \Gamma_j : k2^{-j} \leq \tilde{\delta}_{Q(I)} < (k+1)2^{-j}\}$, $k, j \geq 0$
- $\Gamma_j = \cup_{k \geq 0} \Gamma_{j,k}$
- $\Gamma = \cup_{j \geq 0} \Gamma_j$


T
t3
t2
t1
Ω1
Ω2
I1
I2
I3
I4

Geometría

- Existen c_1 y c_2 tal que si $t > c_1 4^{-j}$ y $I \in \overset{o}{\Gamma}{}_j^{\,2}(t)$, tenemos $\delta(x,t) \geq c_2\sqrt{t}$ para $x \in Q(I)$.

Geometría

- Existen c_1 y c_2 tal que si $t > c_1 4^{-j}$ y $I \in \overset{o}{\Gamma}{}_j^{\,2}(t)$, tenemos $\delta(x,t) \geq c_2\sqrt{t}$ para $x \in Q(I)$.

- Para todo $t > 0$, para todo $I \in \overset{o}{\Gamma}{}_j^{\,1}(t)$ y para todo $x \in Q(I)$ tenemos que $\delta(x,t) \geq \widetilde{\delta}_{Q(I)}$.

Geometría

- Existen c_1 y c_2 tal que si $t > c_1 4^{-j}$ y $I \in \overset{o}{\Gamma}{}_j^{2}(t)$, tenemos $\delta(x,t) \geq c_2\sqrt{t}$ para $x \in Q(I)$.

- Para todo $t > 0$, para todo $I \in \overset{o}{\Gamma}{}_j^{1}(t)$ y para todo $x \in Q(I)$ tenemos que $\delta(x,t) \geq \widetilde{\delta}_{Q(I)}$.

- Existen C_0, C_1 y C_2 dependiendo de D tal que
  1. $\#(\Gamma_j) \leq C_0 2^{jd}$,

Geometría

- Existen c_1 y c_2 tal que si $t > c_1 4^{-j}$ y $I \in \overset{o}{\Gamma}_j^{\,2}(t)$, tenemos $\delta(x,t) \geq c_2\sqrt{t}$ para $x \in Q(I)$.

- Para todo $t > 0$, para todo $I \in \overset{o}{\Gamma}_j^{\,1}(t)$ y para todo $x \in Q(I)$ tenemos que $\delta(x,t) \geq \widetilde{\delta}_{Q(I)}$.

- Existen C_0, C_1 y C_2 dependiendo de D tal que
  1. $\#(\Gamma_j) \leq C_0 2^{jd}$,
  2. $\#(\Gamma_{j,k}) \leq C_1 2^{j(d-1)}$, $j, k \geq 0$,

Geometría

- Existen c_1 y c_2 tal que si $t > c_1 4^{-j}$ y $I \in \overset{o}{\Gamma}_j^{\,2}(t)$, tenemos $\delta(x,t) \geq c_2\sqrt{t}$ para $x \in Q(I)$.

- Para todo $t > 0$, para todo $I \in \overset{o}{\Gamma}_j^{\,1}(t)$ y para todo $x \in Q(I)$ tenemos que $\delta(x,t) \geq \widetilde{\delta}_{Q(I)}$.

- Existen C_0, C_1 y C_2 dependiendo de D tal que
  1. $\#(\Gamma_j) \leq C_0 2^{jd}$,
  2. $\#(\Gamma_{j,k}) \leq C_1 2^{j(d-1)}$, $j, k \geq 0$,
  3. $\Gamma_{j,k} = \varnothing$ for $k > C_2 2^j$

Demostración

- $u \in L_p((0,T); B_p^\lambda(D))$

Demostración

- $u \in L_p((0,T); B_p^\lambda(D))$
- para t fijo, $U(t)(x) = u(x;t) \in B_p^\lambda(D)$

Demostración

- $u \in L_p((0,T); B_p^\lambda(D))$
- para t fijo, $U(t)(x) = u(x;t) \in B_p^\lambda(D)$
- para t fijo, como D es Lipschitz, $V(t)$ extensión de $U(t)$ a $I\!R^d$,

$$\|V(t)\|_{B_p^\lambda(I\!R^d)} \leq C\,\|U(t)\|_{B_p^\lambda(D)}$$

Demostración

- $u \in L_p((0,T); B_p^\lambda(D))$
- para t fijo, $U(t)(x) = u(x;t) \in B_p^\lambda(D)$
- para t fijo, como D es Lipschitz, $V(t)$ extensión de $U(t)$ a $I\!R^d$,

$$\|V(t)\|_{B_p^\lambda(I\!R^d)} \leq C\,\|U(t)\|_{B_p^\lambda(D)}$$

$$\begin{aligned} V(t) &= P_0 V(t) + \sum_{I \in \mathcal{D}^+} \sum_{\psi \in \Psi} \langle V(t), \psi_{I,p'} \rangle\, \psi_{I,p} \\ &= \sum_{I \in \mathcal{D}_0} \langle V(t), \varphi_I \rangle\, \varphi_I + \sum_{I \in \mathcal{D}^+} \sum_{\psi \in \Psi} \langle V(t), \psi_{I,p'} \rangle\, \psi_{I,p} \end{aligned}$$

Demostración

- $u \in L_p((0,T); B_p^\lambda(D))$
- para t fijo, $U(t)(x) = u(x;t) \in B_p^\lambda(D)$
- para t fijo, como D es Lipschitz, $V(t)$ extensión de $U(t)$ a $I\!R^d$,

$$\|V(t)\|_{B_p^\lambda(I\!R^d)} \leq C\,\|U(t)\|_{B_p^\lambda(D)}$$

$$\begin{aligned} V(t) &= P_0 V(t) + \sum_{I\in\mathcal{D}^+}\sum_{\psi\in\Psi} \langle V(t), \psi_{I,p'}\rangle\, \psi_{I,p} \\ &= \sum_{I\in\mathcal{D}_0} \langle V(t), \varphi_I\rangle\, \varphi_I + \sum_{I\in\mathcal{D}^+}\sum_{\psi\in\Psi} \langle V(t), \psi_{I,p'}\rangle\, \psi_{I,p} \end{aligned}$$

$$W(t) = \sum_{I\in\Gamma_0} \langle V(t), \varphi_I\rangle\, \varphi_I + \sum_{I\in\Gamma}\sum_{\psi\in\Psi} \langle V(t), \psi_{I,p'}\rangle\, \psi_{I,p} = W_0(t) + W_1(t)$$

Demostración

- $u \in L_p((0,T); B_p^\lambda(D))$
- para t fijo, $U(t)(x) = u(x;t) \in B_p^\lambda(D)$
- para t fijo, como D es Lipschitz, $V(t)$ extensión de $U(t)$ a $I\!R^d$,

$$\|V(t)\|_{B_p^\lambda(I\!R^d)} \leq C\,\|U(t)\|_{B_p^\lambda(D)}$$

$$\begin{aligned} V(t) &= P_0 V(t) + \sum_{I\in\mathcal{D}^+}\sum_{\psi\in\Psi} \langle V(t), \psi_{I,p'}\rangle\, \psi_{I,p} \\ &= \sum_{I\in\mathcal{D}_0} \langle V(t), \varphi_I\rangle\, \varphi_I + \sum_{I\in\mathcal{D}^+}\sum_{\psi\in\Psi} \langle V(t), \psi_{I,p'}\rangle\, \psi_{I,p} \end{aligned}$$

$$W(t) = \sum_{I\in\Gamma_0} \langle V(t), \varphi_I\rangle\, \varphi_I + \sum_{I\in\Gamma}\sum_{\psi\in\Psi} \langle V(t), \psi_{I,p'}\rangle\, \psi_{I,p} = W_0(t) + W_1(t)$$

- $W(t) \in B_p^\lambda(I\!R^d)$, $W(t) = V(t) = U(t)$ en D

Demostración

$$\|W_0(t)\|_{B^\alpha_\tau(\mathbb{R}^d)} \leq \sum_{I \in \Gamma_0} |\langle V(t), \varphi_I \rangle| \, \|\varphi_I\|_{B^\alpha_\tau(\mathbb{R}^d)}$$

Demostración

$$\begin{aligned}\|W_0(t)\|_{B^\alpha_\tau(\mathbb{R}^d)} &\le \sum_{I\in\Gamma_0} |\langle V(t),\varphi_I\rangle|\,\|\varphi_I\|_{B^\alpha_\tau(\mathbb{R}^d)} \\ &\le \|V(t)\|_{L_p(\mathbb{R}^d)} \sum_{I\in\Gamma_0} \|\varphi_I\|_{L_{p'}(\mathbb{R}^d)}\,\|\varphi_I\|_{B^\alpha_\tau(\mathbb{R}^d)}\end{aligned}$$

Demostración

$$
\begin{aligned}
\|W_0(t)\|_{B^{\alpha}_{\tau}(\mathbb{R}^d)} &\le \sum_{I\in\Gamma_0} |\langle V(t), \varphi_I\rangle| \, \|\varphi_I\|_{B^{\alpha}_{\tau}(\mathbb{R}^d)} \\
&\le \|V(t)\|_{L_p(\mathbb{R}^d)} \sum_{I\in\Gamma_0} \|\varphi_I\|_{L_{p'}(\mathbb{R}^d)} \, \|\varphi_I\|_{B^{\alpha}_{\tau}(\mathbb{R}^d)} \\
&\le C \, \|V(t)\|_{B^{\lambda}_{p}(\mathbb{R}^d)}
\end{aligned}
$$

Demostración

$$
\begin{aligned}
\|W_0(t)\|_{B^\alpha_\tau(\mathbb{R}^d)} &\leq \sum_{I\in\Gamma_0} |\langle V(t), \varphi_I\rangle| \, \|\varphi_I\|_{B^\alpha_\tau(\mathbb{R}^d)} \\
&\leq \|V(t)\|_{L_p(\mathbb{R}^d)} \sum_{I\in\Gamma_0} \|\varphi_I\|_{L_{p'}(\mathbb{R}^d)} \, \|\varphi_I\|_{B^\alpha_\tau(\mathbb{R}^d)} \\
&\leq C \, \|V(t)\|_{B^\lambda_p(\mathbb{R}^d)} \\
&\leq C \, \|U(t)\|_{B^\lambda_p(D)}
\end{aligned}
$$

Demostración

$$\begin{aligned}\|W_0(t)\|_{B^\alpha_\tau(\mathbb{R}^d)} &\le \sum_{I\in\Gamma_0} |\langle V(t),\varphi_I\rangle|\,\|\varphi_I\|_{B^\alpha_\tau(\mathbb{R}^d)}\\ &\le \|V(t)\|_{L_p(\mathbb{R}^d)} \sum_{I\in\Gamma_0} \|\varphi_I\|_{L_{p'}(\mathbb{R}^d)}\,\|\varphi_I\|_{B^\alpha_\tau(\mathbb{R}^d)}\\ &\le C\,\|V(t)\|_{B^\lambda_p(\mathbb{R}^d)}\\ &\le C\,\|U(t)\|_{B^\lambda_p(D)}\end{aligned}$$

$$\int_0^T \|W_0(t)\|^\tau_{B^\alpha_\tau(\mathbb{R}^d)}\,dt \le C\int_0^T \|U(t)\|^\tau_{B^\lambda_p(D)}\,dt$$

Demostración

$$\begin{aligned}\|W_0(t)\|_{B^\alpha_\tau(\mathbb{R}^d)} &\leq \sum_{I\in\Gamma_0} |\langle V(t),\varphi_I\rangle|\,\|\varphi_I\|_{B^\alpha_\tau(\mathbb{R}^d)} \\ &\leq \|V(t)\|_{L_p(\mathbb{R}^d)} \sum_{I\in\Gamma_0} \|\varphi_I\|_{L_{p'}(\mathbb{R}^d)}\,\|\varphi_I\|_{B^\alpha_\tau(\mathbb{R}^d)} \\ &\leq C\,\|V(t)\|_{B^\lambda_p(\mathbb{R}^d)} \\ &\leq C\,\|U(t)\|_{B^\lambda_p(D)}\end{aligned}$$

$$\int_0^T \|W_0(t)\|^\tau_{B^\alpha_\tau(\mathbb{R}^d)}\,dt \leq C\int_0^T \|U(t)\|^\tau_{B^\lambda_p(D)}\,dt$$

como $\tau < p$, aplicando la desiguadad de Hölder $\frac{p}{\tau}$

$$\int_0^T \|W_0(t)\|^\tau_{B^\alpha_\tau(\mathbb{R}^d)}\,dt \leq CT^{\frac{p-\tau}{p}}\left(\int_0^T \|U(t)\|^p_{B^\lambda_p(D)}\,dt\right)^{\frac{\tau}{p}}$$

Demostración

$$
\begin{aligned}
\|W_0(t)\|_{B^\alpha_\tau(\mathbb{R}^d)} &\leq \sum_{I\in\Gamma_0} |\langle V(t),\varphi_I\rangle| \, \|\varphi_I\|_{B^\alpha_\tau(\mathbb{R}^d)} \\
&\leq \|V(t)\|_{L_p(\mathbb{R}^d)} \sum_{I\in\Gamma_0} \|\varphi_I\|_{L_{p'}(\mathbb{R}^d)} \|\varphi_I\|_{B^\alpha_\tau(\mathbb{R}^d)} \\
&\leq C\,\|V(t)\|_{B^\lambda_p(\mathbb{R}^d)} \\
&\leq C\,\|U(t)\|_{B^\lambda_p(D)}
\end{aligned}
$$

$$
\int_0^T \|W_0(t)\|^\tau_{B^\alpha_\tau(\mathbb{R}^d)}\,dt \leq C\int_0^T \|U(t)\|^\tau_{B^\lambda_p(D)}\,dt
$$

como $\tau < p$, aplicando la desiguadad de Hölder $\frac{p}{\tau}$

$$
\int_0^T \|W_0(t)\|^\tau_{B^\alpha_\tau(\mathbb{R}^d)}\,dt \leq CT^{\frac{p-\tau}{p}}\left(\int_0^T \|U(t)\|^p_{B^\lambda_p(D)}\,dt\right)^{\frac{\tau}{p}}
$$

es finita pues $u \in L_p((0,T);B^\lambda_p(D))$.

Demostración

Para ver $\int_0^T \|W_1(t)\|^\tau_{B^\alpha_\tau(\mathbb{R}^d)}\, dt < \infty$ basta ver

$$\int_0^T \sum_{I\in\Gamma} \sum_{\psi\in\Psi} |\langle V(t), \psi_{I,p'}\rangle|^\tau \, dt < \infty$$

Demostración

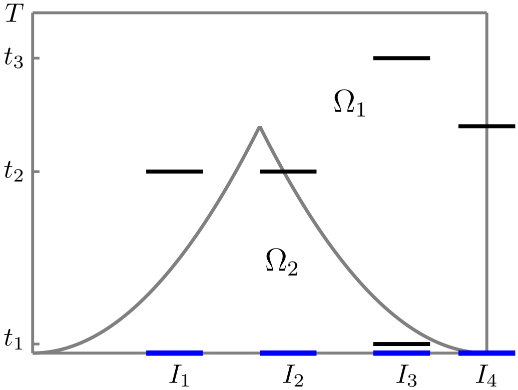
Para ver $\int_0^T \|W_1(t)\|^\tau_{B^\alpha_\tau(\mathbb{R}^d)}\, dt < \infty$ basta ver

$$\int_0^T \sum_{I\in\Gamma} \sum_{\psi\in\Psi} |\langle V(t), \psi_{I,p'}\rangle|^\tau \, dt < \infty$$

$$\begin{aligned}
\int_0^T \sum_{I\in\Gamma} \sum_{\psi\in\Psi} |\langle V(t), \psi_{I,p'}\rangle|^\tau \, dt &= \sum_{j=0}^{\infty} \int_0^T \sum_{I\in\Gamma_j} \sum_{\psi\in\Psi} |\langle V(t), \psi_{I,p'}\rangle|^\tau \, dt \\
&= \sum_{j=0}^{\infty} \int_0^{c_1 4^{-j}} \sum_{I\in\Gamma_j} \sum_{\psi\in\Psi} |\langle V(t), \psi_{I,p'}\rangle|^\tau \, dt \\
&\quad + \sum_{j=0}^{\infty} \int_{c_1 4^{-j}}^T \sum_{I\in\Gamma_j} \sum_{\psi\in\Psi} |\langle V(t), \psi_{I,p'}\rangle|^\tau \, dt \\
&= A + B
\end{aligned}$$

Demostración

$$
\begin{aligned}
B &= \sum_{j=0}^{\infty} \int_{c_1 4^{-j}}^{T} \sum_{I \in \Gamma_{j,0}} \sum_{\psi \in \Psi} |\langle V(t), \psi_{I,p'} \rangle|^{\tau} \, dt \\
&\quad + \sum_{j=0}^{\infty} \int_{c_1 4^{-j}}^{T} \sum_{I \in \overset{o}{\Gamma}{}_j^{1}(t)} \sum_{\psi \in \Psi} |\langle V(t), \psi_{I,p'} \rangle|^{\tau} \, dt \\
&\quad + \sum_{j=0}^{\infty} \int_{c_1 4^{-j}}^{T} \sum_{I \in \overset{o}{\Gamma}{}_j^{2}(t)} \sum_{\psi \in \Psi} |\langle V(t), \psi_{I,p'} \rangle|^{\tau} \, dt \\
&= B_0 + B_1 + B_2
\end{aligned}
$$



T
t3
t2
t1
Ω1
Ω2
I1
I2
I3
I4