Global $W^{2,p}$ estimates for nondivergence elliptic operators with potentials satisfying a reverse Hölder condition

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Schrödinger type operators

Let us consider the linear, second order elliptic operator

$$Lu \equiv Au + Vu \equiv -a_{ij} u_{x_i x_j} + Vu$$

where (for $i, j = 1, 2, ..., n$)$a_{ij} \in L^{\infty}(\mathbb{R}^n)$, $a_{ij} = a_{ji}$,

$$\mu |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \frac{1}{\mu} |\xi|^2 \quad \forall x, \xi \in \mathbb{R}^n, \text{ for some } \mu > 0, \qquad (1)$$

$$a_{ij} \in VMO(\mathbb{R}^n) \qquad (2)$$

which means that for $i, j = 1, 2, ..., n$

$$\eta_{ij}(r) = \sup_{\rho \leq r} \sup_{x \in \mathbb{R}^n} \left(\frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} \left| a_{ij}(y) - a_{ij}^B \right| dy \right)$$

vanishes for $r \to 0^+$.
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The potential V

As to the potential V, we assume that it is not identically zero and that

$$V \in B_q \text{ for some } q \geq \frac{n}{2}, \tag{3}$$

which by definition means that $V \in L^q_{loc}$, $V \geq 0$ and there exists a constant $C > 0$ such that the *reverse Hölder inequality*

$$\left(\frac{1}{|B|}\int_B V(x)^q\,dx\right)^{1/q} \leq C\left(\frac{1}{|B|}\int_B V(x)\,dx\right)$$

holds for every ball B in $\mathbb{R}^n$.

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The B_q Condition

- $V \in B_q$ also implies $V \in B_{q+\varepsilon}$ for some $\varepsilon > 0$.
- The measure $d\mu(y) = V(y)\, dy$ is doubling

A model example:

$$V(x) = |x|^2 .$$

More generally,if V is any nonnegative polynomial, then V satisfies the stronger condition

$$\max_{x \in B} V(x) \leq C \frac{1}{|B|} \int_B V(x)\, dx \ ,$$

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Global a piori estimates

We are mainly interested in proving global a priori L^p estimates of the kind

Theorem

Under the assumptions (1), (2), (3), for every $p \in (1, q]$ there exists a constant $C > 0$ such that

$$\|u\|_{W^{2,p}(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)} \leq C \left\{ \|Lu\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} \right\} \qquad (4)$$

for any $u \in C_0^\infty(\mathbb{R}^n)$. The constant C depends on n, p, q, the ellipticity constant μ, the VMO moduli of the leading coefficients, and the B_q constant of V.

The bound (4) immediately extends to all functions $u \in W_V^{2,p}(\mathbb{R}^n)$, the closure of $C_0^\infty(\mathbb{R}^n)$ in the norm

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Previous Results

When A is the Laplacian, these bounds have been proved by

- Thangavelu, for $V(x) = |x|^2$ (Hermite operator)
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Strategy to prove global a piori estimates

From $Lu = Au + Vu$, we can write

$$Au = Lu - Vu.$$

By CFL, we get

$$\begin{aligned} \left\|D^2 u\right\|_{L^p(\mathbb{R}^n)} &\leq C\left\{\|Au\|_{L^p(\mathbb{R}^n)}\right\} \\ &\leq C\left\{\|Lu\|_{L^p(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)}\right\} \end{aligned}$$

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Local Results

More precisely, we need the following Local Results:

Theorem

Main Local Theorem.

Under the assumptions (1), (2), (3), for any $p \in (1, q]$ there exist positive constants C, r such that for any $z_0 \in \mathbb{R}^n$, $u \in C_0^\infty(B_r(z_0))$

$$\|Vu\|_{L^p(B_r(z_0))} \leq C \|Lu\|_{L^p(B_r(z_0))}.$$

we also use the following basic result proved by Chiarenza-Frasca-Longo

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(CFL Local)

Under the assumptions (1), (2), for any $p \in (1, \infty)$ there exist positive constants C, r such that for any $z_0 \in \mathbb{R}^n$, $u \in C_0^\infty(B_r(z_0))$

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Proof of the Main Global a priori L^p estimates by Theorems Main Local and CFL Local

Proof.

Let $\{\phi_i\}_{i=1}^{\infty}$ be a partition of unity of non negative functions in $\mathbb{R}^n$ such that $\phi_i \in C_0^\infty (B(z_i, r))$ with r as in theorem "Main Local" and such that the family of balls $B_i = B(z_i, r)$ has the finite overlapping property.

□

Proof of the Main Global a priori L^p estimates by Theorems Main Local and CFL Local

Proof.

$$
\begin{aligned}
\|Vu\|^p_{L^p(\mathbb{R}^n)} &= \left\| \sum_i V\phi_i u \right\|^p_{L^p(\mathbb{R}^n)} \\
&\leq C \sum_i \|V\phi_i u\|^p_{L^p(B(z_i,r))} \leq C \sum_i \|L(\phi_i u)\|^p_{L^p(B(z_i,r))} \\
&\leq C \sum_i \left\{ \|Lu\|^p_{L^p(B(z_i,r))} + \|Du\|^p_{L^p(B(z_i,r))} + \|u\|^p_{L^p(B(z_i,r))} \right\} \\
&\leq C \left\{ \|Lu\|^p_{L^p(\mathbb{R}^n)} + \|Du\|^p_{L^p(\mathbb{R}^n)} + \|u\|^p_{L^p(\mathbb{R}^n)} \right\} \\
&\leq C \left\{ \|Lu\|_{L^p(\mathbb{R}^n)} + \|Du\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} \right\}^p . \qquad (5)
\end{aligned}
$$

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Proof of the Main Global a priori L^p estimates by Theorems Main Local and CFL Local

Proof.

Analogously, Theorem "CFL Local" implies

$$\left\|D^2 u\right\|_{L^p(\mathbb{R}^n)} \leq C\left\{\|Au\|_{L^p(\mathbb{R}^n)} + \|Du\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}\right\}$$
$$\leq C\left\{\|Lu\|_{L^p(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)} + \|Du\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}\right\}$$

which, together with (**??**) gives

$$\|u\|_{W^{2,p}(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)} \leq C\left\{\|Lu\|_{L^p(\mathbb{R}^n)} + \|Du\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}\right\}.$$

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Proof.

Then the classical interpolation inequality

$$\|Du\|_{L^p(\mathbb{R}^n)} \leq \varepsilon \left\|D^2 u\right\|_{L^p(\mathbb{R}^n)} + \frac{C}{\varepsilon} \|u\|_{L^p(\mathbb{R}^n)}$$

allows to write

$$\|u\|_{W^{2,p}(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)} \leq C \left\{ \|Lu\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} \right\}.$$

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Proof Main Local Theorem

To prove the Main Local theorem, we pick a ball $B_r(z_0)$ with r to be chosen later and a point $x_0 \in B_r(z_0)$.

We freeze the coefficients of A at x_0, getting the operator

$$L_0 u = -a_{ij}(x_0)\, u_{x_i x_j} + V(x)\, u.$$

This operator can be rewritten in divergence form

$$L_0 u = -\left(a_{ij}(x_0)\, u_{x_i}\right)_{x_j} + V(x)\, u,$$

which allows us to apply the results proved by Dziubanski for this type operator (and in divergence form) to deduce

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$$L_0 u = -\left(a_{ij}(x_0)\, u_{x_i}\right)_{x_j} + V(x)\, u,$$

which allows us to apply the results proved by Dziubanski for this type operator (and in divergence form) to deduce

Continuing the proof

For any $u \in C_0^\infty(B_r(z_0))$, $x \in B_r(z_0)$, we can write:

$$u(x) = \int \Gamma(x_0; x, y) L_0 u(y)\, dy =$$

$$= \int \Gamma(x_0; x, y) Lu(y)\, dy + \int \Gamma(x_0; x, y) [L_0 u(y) - Lu(y)]\, dy =$$

$$= \int \Gamma(x_0; x, y) Lu(y)\, dy + \int \Gamma(x_0; x, y) [A_0 u(y) - Au(y)]\, dy.$$

Where $\Gamma(x_0; x, y)$ is given in

Theorem

(Dziubanski)

The operator L_0 has a fundamental solution $\Gamma(x_0; x, y)$ satisfying the following bound: for any positive integer k there exists a constant c_k (independent of x_0) such that

$$\Gamma(x_0; x, y) \leq \frac{c_k}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^k} \cdot \frac{1}{|x-y|^{n-2}} \text{ for any } x, y \in \mathbb{R}^n, x \neq y$$

where $\rho(x)$ is the "critical radius" associated to V, defined by:

$$\rho(x) = \sup\left\{ r > 0 : \frac{r^2}{|B(x,r)|} \int_{B(x,r)} V(y)\, dy \leq 1 \right\}.$$

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Continuing the proof

Unfreezing now the coefficients, letting $x_0 = x$, we get the representation formula for Vu:

$$u(x) = \int \Gamma(x; x, y) \, Lu(y) \, dy + \sum_{i,j=1}^{n} \int \Gamma(x; x, y) \left[a_{ij}(y) - a_{ij}(x) \right] u_{x_i x_j}(y) \, dy$$

which allows us to write the following pointwise bound, for every positive integer k:

$$|V(x) u(x)| \leq c_k V(x) \int \frac{1}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^k} \cdot \frac{1}{|x-y|^{n-2}} \cdot \left\{ |Lu(y)| + \sum_{i,j=1}^{n} |a_{ij}(y) - a_{ij}(x)| \left| u_{x_i x_j}(y) \right| \right\} dy$$

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for any $x \in \mathbb{R}^n$ and any positive integer k.

Continuing the Proof MLT

Let us introduce the integral operators:

$$S_k f(x) = V(x) \int \frac{1}{\left(1+\frac{|x-y|}{\rho(x)}\right)^k} \cdot \frac{1}{|x-y|^{n-2}} f(y)\, dy$$

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for $a \in L^\infty \cap VMO(\mathbb{R}^n)$.
So that our representation formula can be written in compact form as

$$|Vu(x)| \leq c_k S_k(|Lu|)(x) + \sum_{i,j=1}^{n} S_{k,a_{ij}}\left(\left|u_{x_i x_j}\right|\right)(x). \tag{5}$$

This formula involves suitable (nonsingular) integral operators with positive kernels, applied to Lu, and their commutators, applied to the second order derivatives of u.

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Required to prove MLT

We will prove that for any $p \in (1, q]$ and for k large enough

$$\|S_k f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \tag{6}$$

and that for any $\varepsilon > 0$ there exists r, depending on the VMO modulus of the function a, such that

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Finish proof Main Local Theorem

Now, by (5), (6), (7) and the Theorem "CFL Local" , for any $u \in C_0^\infty(B_r(z_0))$, r small enough, we have

$$\|Vu\|_{L^p} \leq C\|Lu\|_{L^p} + \varepsilon \left\|u_{x_i x_j}\right\|_{L^p}$$
$$\leq C\|Lu\|_{L^p} + C\varepsilon\|Au\|_{L^p} \leq (C + C\varepsilon)\|Lu\|_{L^p} + C\varepsilon\|Vu\|_{L^p}$$

whence, the proof of the Main Local Theorem is finished.

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whence, the proof of the Main Local Theorem is finished.

The L^p estimates $S_k, S_{k,a}$

It is more convenient to consider the transposed operators:

$$S_k^* f(x) = \int \frac{V(y)}{\left(1 + \frac{|x-y|}{\rho(y)}\right)^k} \cdot \frac{1}{|x-y|^{n-2}} f(y)\, dy;$$

$$S_{k,a}^* f(x) = \int \frac{V(y)}{\left(1 + \frac{|x-y|}{\rho(y)}\right)^k} \cdot \frac{1}{|x-y|^{n-2}} |a(y) - a(x)|\, f(y)\, dy.$$

Theorem

For k large enough, the operator S_k^* is continuous on $L^p(\mathbb{R}^n)$ for $p \in [q', \infty]$ (where q' is the conjugate exponent of q, and $V \in B_q$).

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For k large enough, the operator $S_{k,a}^*$ is continuous on $L^p(\mathbb{R}^n)$ for $p \in [q', \infty)$ and for any $\varepsilon > 0$ there exists $r > 0$, depending on the VMO modulus of a, such that

$$\left\| S_{k,a}^* f \right\|_{L^p(B_r(z_0))} \leq \varepsilon \left\| f \right\|_{L^p(B_r(z_0))}.$$

Proof S_k^* Theorem

We prove the following pointwise bound

$$S_k^* f(x) \leq C \ M_{q'} f \ (x) \tag{8}$$

for any $x \in \mathbb{R}^n, f \in L^p(\mathbb{R}^n), f \geq 0$, where $M_{q'} f$ is the maximal function of exponent q', i.e.

$$M_{q'} f(x) = \sup_{B \ni x} \left(\frac{1}{|B|} \int_B f(y)^{q'} \, dy \right)^{1/q'} .$$

Proof $S^*_{k,a}$ Theorem

We will deduce the estimate for $S^*_{k,a}$from an abstract result.

Theorem

Let $W(x,y)$ be a non-negative kernel satisfying $H_1(q)$ for some $q>1$ and such that the integral operator

$$Tf(x) = \int_{\mathbb{R}^n} W(x,y)\, f(y)\, dy$$

is continuous on $L^p(\mathbb{R}^n)$ for any $p \in (q',\infty)$. Then for $b \in BMO(\mathbb{R}^n)$ the operator ("positive commutator")

$$T_b f(x) = \int_{\mathbb{R}^n} |b(x) - b(y)|\, W(x,y)\, f(y)\, dy$$

is bounded on $L^p(\mathbb{R}^n)$ for any $p \in (q',\infty)$, and

$$\|T_b f\|_p \le C \|b\|_{BMO} \|f\|_p$$

Proof Abstract Theorem

We will prove the following point-wise inequality: for any $s > q'$, there exists a constant C such that

$$(T_b f)^{\#}(z) \leqslant C \|b\|_{BMO} \left[M_s(Tf)(z) + (M_s f)(z) \right], \tag{9}$$

where

$$g^{\#}(z) = \sup_{B \ni z} \frac{1}{|B|} \int_B |g(x) - g_B| \, dx$$

is the sharp maximal function. This, by Fefferman-Stein's inequality together with the maximal theorem, will imply our result.

Hörmander's condition of order q

Definition

We say that the kernel $W(x,y)$ satisfies Hörmander's condition of order q in the first variable, briefly $W \in H_1(q)$ if:
there exists a constant C such that for any $r > 0$ and $x, x_0 \in \mathbb{R}^n$ such that $|x - x_0| \leq r$, the following inequality holds

$$\sum_{j=1}^{\infty} j\left(2^j r\right)^{n/q'} \left(\int_{2^j r \leq |x_0 - y| \leq 2^{j+1} r} |w(x,y) - w(x_0,y)|^q \, dy\right)^{1/q} \leq C.$$

Lemma

The kernel

$$w(x,y) = \frac{V(y)}{\left(1 + \frac{|x-y|}{\rho(y)}\right)^k} \cdot \frac{1}{|x-y|^{n-2}} \qquad (10)$$

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Existence and uniqueness results

Finally, we apply such a priori estimates to derive some global existence and uniqueness results under the additional assumptions on V

$$V(x) \geq \delta > 0 \text{ for any } x \in \mathbb{R}^n \tag{11}$$

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