Resultados parciales de una conjetura de Muckenhoupt y Wheeden

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Based in a joint work with Andrei Lerner y Carlos Pérez

Main objects

1. Hardy-Littlewood maximal operator

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2. Calderón-Zygmund operators By a Calderón-Zygmund operator we mean a bounded operator on $L^2(\mathbb{R}^n)$, and whose distributional kernel K coincides away from the diagonal $x = y$ in $\mathbb{R}^n \times \mathbb{R}^n$ with a function K satisfying the size estimate

$$|K(x,y)| \leq \frac{c}{|x-y|^n}$$

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \leq c \frac{|x-z|^\varepsilon}{|x-y|^{n+\varepsilon}},$$

for some $\varepsilon > 0$, and if whenever $2|x-z| < |x-y|$. So that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy,$$

whenever $f \in C_0^\infty(\mathbb{R}^n)$ and $f \in C_0^\infty(\mathbb{R}^n)$.

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A_1 weights

We say that ω satisfies the A_1 condition if there exists $c > 0$ such that

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- $(Mf)^{\alpha} \in A_1$ for $0 < \alpha < 1$ (R. Coifman and R. Rochberg, 1980).

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For any weight ω and any function f,

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Theorem (R. Hunt, B. Muckenhoupt and R. Wheeden, 1973; R. Coifman and C. Fefferman, 1974)

Let T be a Calderón-Zygmund operator. If $\omega \in A_1$, then

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- The weak M-W conjecture is open, in general, even for the Hilbert transform

$$Hf(x) = \text{P.V.} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy.$$

Unweighted weak type $(1,1)$ of T

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Sketch of the proof (Calderón-Zygmund decomposition):

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↑ Chebyshev's inequality

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$$|g| \le c\alpha$$

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definition of g

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For any weight ω and any function f,

$$\int_{\mathbb{R}^n} |Tf|^p \omega dx \leq c \int_{\mathbb{R}^n} |f|^p M^{[p]+1} \omega dx \quad (p > 1),$$

where $[p]$ is the integer part of p.

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- The method of the proof gives $c_\varepsilon \approx c^{1/\varepsilon}$. This does not allow to bound $c_\varepsilon\|\omega\|_{A_1}^{1+\varepsilon}$ by $\|\omega\|_{A_1}(\log\|\omega\|_{A_1})^\gamma$ for any $\gamma > 0$.

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- A stronger inequality

$$\Big(\int_{\mathbb{R}^n}|Tf|^2\omega dx\Big)^{1/2} \leq c\Big(\int_{\mathbb{R}^n}|f|^2 M^2\omega dx\Big)^{1/2}$$

does not hold (J.M. Wilson-C. Pérez).

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- The proof is based on square function estimates. In particular, a deep Chang-Wilson-Wolff theorem is used.

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- The method of the proof shows that $\alpha = 1$ for all $p \geq 2$; however, for $1 < p < 2$ it gives only that $\alpha \leq 1/2 + 1/p$.

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- A question of interest is also the sharp dependence of c on p.
- If the weak M-W conjecture holds, then

$$\|Tf\|_{L^p(\omega)} \le c\frac{p^2}{p-1}\|\omega\|_{A_1}\|f\|_{L^p(\omega)} \quad (p > 1).$$

Main Results

Theorem (L-O-P, 2008)

For any Calderón-Zygmund operator T*,*

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Theorem (L-O-P, 2008)

Let $\varphi(t) = t\log(1+t)$. Then, for any Calderón-Zygmund operator T,

$$\omega\{x\in\mathbb{R}^n : |Tf(x)|>\alpha\} \le \frac{c}{\alpha}\varphi(\|\omega\|_{A_1})\int_{\mathbb{R}^n}|f|\omega\,dx \quad (\alpha>0).$$

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- That Theorem improves a previous one:

Theorem (L-O-P, 2007 (IMRN 2008))

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Two key estimates

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Let $\omega \in A_1$, and let $r_\omega = 1 + \frac{1}{2^{n+1}\|\omega\|_{A_1}}$. Then

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- In particular, setting $r = r_\omega$, we get

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- Next, applying the Calderón-Zygmund method and the above estimates to the "good" part, we obtain

$$\omega\{x : |Tf(x)| > \alpha\} \leq c\left(\frac{\|\omega\|_{A_1}}{p-1}\right)^p \frac{1}{\alpha}\int_{\mathbb{R}^n} |f|\omega dx.$$

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- It remains to optimize this inequality with respect to p. Setting $p = 1 + \frac{1}{\log(1+\|\omega\|_{A_1})}$ gives

$$\omega\{x : |Tf(x)| > \alpha\} \leq c\|\omega\|_{A_1}\log(1+\|\omega\|_{A_1})\frac{1}{\alpha}\int_{\mathbb{R}^n} |f|\omega dx.$$

Main ideas used in the proof

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- Restatement by duality:

$$\|T^*f\|_{L^{p'}\left((M_r\omega)^{1-p'}\right)} \leq c\frac{p^2}{p-1}\frac{1}{r-1}\|f\|_{L^{p'}(\omega^{1-p'})}.$$

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$$\|M\left((f\omega^{-1/p})^{(pr)'}\right)\|_{L^{\frac{p'}{(pr)'}}}^{\frac{1}{(pr)'}} \leq c\frac{p}{r-1}\|f\|_{L^{p'}(\omega^{1-p'})}.$$

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- $u = (M_r\omega)^{1-p'} \in A_\infty$. The standard method of the proof applying to u gives $2^{p'}$ instead of p'.
- Previously, we obtained $p' log(\frac{1}{p-1})$

Coifman-type estimate

$$\|T^*f\|_{L^{p'}\left((M_r\omega)^{1-p'}\right)} \leq cp'\|Mf\|_{L^{p'}\left((M_r\omega)^{1-p'}\right)}.$$

- By duality,

$$\|T^*f\|_{L^{p'}\left((M_r\omega)^{1-p'}\right)} = \sup_{\|h\|_{L^p(M_r\omega)}=1} \int_{\mathbb{R}^n} |T^*f| h\, dx,$$

so we have to show that

$$\sup_{\|h\|_{L^p(M_r\omega)}=1} \|T^*f\|_{L^1(h)} \leq cp'\|Mf\|_{L^{p'}\left((M_r\omega)^{1-p'}\right)}.$$

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- An improved Coifman-type estimate: if $u \in A_3$, then

$$\|T^*f\|_{L^1(u)} \le c\|u\|_{A_3}\|Mf\|_{L^1(u)},$$

where

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- The proof is based on results due to S. Buckley (1993) and R. Fefferman and J. Pipher (1997).

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$$\sup_{\|h\|_{L^p(M_r\omega)}=1} \|T^* f\|_{L^1(h)} \leq cp' \|Mf\|_{L^{p'}\left((M_r\omega)^{1-p'}\right)}.$$

$$\|T^* f\|_{L^1(u)} \leq c\|u\|_{A_3} \|Mf\|_{L^1(u)}.$$

- There exists an operator R satisfying

$$h \leq R(h), \quad \|R\|_{L^p(M_r\omega)} \leq 2, \quad \|R(h)(M_r\omega)^{1/p}\|_{A_1} \leq cp'.$$

In particular, the last property yields $\|Rh\|_{A_3} \leq cp'$.

Coifman-type estimate

$$\sup_{\|h\|_{L^p(M_r\omega)}=1} \|T^* f\|_{L^1(h)} \le c p' \|Mf\|_{L^{p'}\left((M_r\omega)^{1-p'}\right)}.$$

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- R is constructed by means of the Rubio de Francia method:

$$S(h) = \frac{M\left(h(M_r\omega)^{1/p}\right)}{(M_r\omega)^{1/p}} \quad \text{and} \quad R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k(h)}{(\|S\|_{L^p(M_r\omega)})^k}.$$

Coifman-type estimate

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- By the above estimates and Hölder's inequality,

$$\begin{aligned}\|T^*f\|_{L^1(h)} \leq \|T^*f\|_{L^1(Rh)} &\leq c\|Rh\|_{A_3}\|Mf\|_{L^1(Rh)}\\ &\leq cp'\|Mf\|_{L^{p'}\left((M_r\omega)^{1-p'}\right)}.\end{aligned}$$

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Closely related open questions II

- Consider the following version of the weak M-W conjecture

$$\omega\{x \in \mathbb{R} : Hf(x) > 1\} \leq c\|\omega\|_{A_1}\|f\|_{L^1(\omega)}$$

in the case when $f = \sum_{k=1}^{n} \delta_{t_k}$, where $t_1 < t_2 < \cdots < t_n$.

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- We get

$$\omega\left\{x \in \mathbb{R} : \sum_{k=1}^{n} \frac{1}{x - t_k} > 1\right\} \leq c\|\omega\|_{A_1} \sum_{k=1}^{n} \omega(t_k).$$

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- $\omega \equiv 1$ [G. Boole (1857), L.H. Loomis (1946)]:
  Let $h(r_j) = 1$. Then $\{h > 1\} = \cup_{k=1}^{n}(t_k, r_k)$, where $t_k < r_k < t_{k+1}$ and $\sum_{k=1}^{n}(r_k - t_k) = n$. Hence, $|\{h > 1\}| = n$.
- Is it possible to extend this approach to the case $\omega \in A_1$?

Closely related open questions III: A_2 problem

1. $$\|w\|_{A_p} \equiv \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty.$$

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2. It is a difficult open problem whether a Calderón-Zygmund operator T satisfies the following sharp inequality with respect to $\|w\|_{A_p}$:

$$\|Tf\|_{L^p(w)} \leq c \|w\|_{A_p}^{\max\left\{1, \frac{1}{p-1}\right\}} \|f\|_{L^p(w)} \quad (1 < p < \infty).$$

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Corollary (L-O-P 2008))

Let $1 < p < \infty$ and let T be a Calderón-Zygmund operator. Also let $w \in A_p$, then

$$\|Tf\|_{L^{p,\infty}(w)} \le c\|w\|_{A_p}(1 + \log \|w\|_{A_p})\|f\|_{L^p(w)},$$