Convergence of Adaptive Finite Elements

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6. March 2009, Santa Fe

Outline

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Continuous Problem

Consider a linear, elliptic PDE over a polygonal, bounded domain $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$)

$$\mathcal{L}u = f \quad \text{in } \Omega, \qquad \text{and some boundary conditions on } \partial\Omega$$

in variational formulation:

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where

1. $\mathbb{V}$ is an Hilbert space, for instance $H_0^1(\Omega)$, $H^1(\Omega)/\mathbb{R}$, $H_0^1(\Omega;\mathbb{R}^d) \times L_2(\Omega)/\mathbb{R}$, $H_0(\mathrm{div};\mathbb{R}^d)$, $H_0(\mathrm{curl};\mathbb{R}^d)$;
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2. $f \in \mathbb{V}^*$ an element of the dual space,
3. $\mathcal{B}\colon \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ is a continuous bilinear form that satisfies an inf-sup condition:

$$|\mathcal{B}[v, w]| \le C^* \|v\|_{\mathbb{V}} \, \|w\|_{\mathbb{V}} \qquad \forall v, w \in \mathbb{V},$$

$$\inf_{v\in\mathbb{V}} \sup_{w\in\mathbb{V}} \frac{\mathcal{B}[v, w]}{\|v\|_{\mathbb{V}}\|w\|_{\mathbb{V}}} = c_* > 0,$$

$$\forall w \in \mathbb{V} \setminus \{0\} \;\; \exists v \in \mathbb{V}: \qquad \mathcal{B}[v, w] \neq 0.$$

Existence and Uniqueness [Nečas '62]

Theorem (Existence and Uniqueness). Problem (P) has for any $f \in \mathbb{V}^*$ a unique solution $u \in \mathbb{V}$ **if and only if** the bilinear form $\mathcal{B}$ is continuous and satisfies the inf-sup condition

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2. Existence and uniqueness for coercive $\mathcal{B}$, i. e.,

$$\mathcal{B}[v,v] \ge c_* \|v\|_{\mathbb{V}}^2 \qquad \forall v \in \mathbb{V},$$

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   follows from Lax-Milgram Theorem ['54]. Coercivity implies the inf-sup and is inherited to any subspace of $\mathbb{V}$ with the same constant c_*.
3. The inf-sup condition is more general than coercivity but, in general, the inf-sup condition is not valid on subspaces of $\mathbb{V}$!

Example: Linear Elliptic PDE

Poisson problem: For given $f \in L_2(\Omega)$ solve for u such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

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Here, $\mathbb{V} := H_0^1(\Omega)$, $\|\cdot\|_{\mathbb{V}} = \|\cdot\|_{H^1(\Omega)}$, and for $u, v \in \mathbb{V}$ set

$$\begin{aligned} \mathcal{B}[u,v] &:= \int_\Omega \nabla u \cdot \nabla v \, dx, \\ \langle f, v \rangle &:= \int_\Omega f \, v \, dx. \end{aligned}$$

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$\mathcal{B}$ is continuous and coercive, i. e.,

$$\mathcal{B}[v,v] \geq c_* \|v\|_{H^1(\Omega)}^2 \qquad \forall v \in H_0^1(\Omega),$$

thanks to the Poincaré-Friedrichs inequality

$$\|v\|_{L_2(\Omega)} \leq C \|\nabla v\|_{L_2(\Omega)} \qquad \forall v \in H_0^1(\Omega).$$

Example: Linear Saddle-Point Problem

Stokes problem: For given $\boldsymbol{f} \in L_2(\Omega; \mathbb{R}^d)$ solve for velocity $\boldsymbol{u}$ and pressure p such that

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega. \end{aligned}$$

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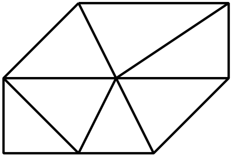
Here, $\mathbb{V} = H_0^1(\Omega; \mathbb{R}^d) \times L_2(\Omega)/\mathbb{R}$ and for $u = (\boldsymbol{u}, p), v = (\boldsymbol{v}, q) \in \mathbb{V}$ set

$$\begin{aligned} \mathcal{B}[u, q] &:= \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, dx - \int_\Omega p \, \nabla \cdot \boldsymbol{v} \, dx - \int_\Omega \nabla \cdot \boldsymbol{u} \, q \, dx, \\ \langle f, v \rangle &:= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx. \end{aligned}$$

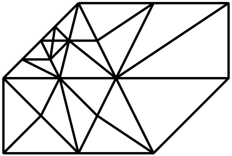
$\mathcal{B}$ is continuous and fulfills the inf-sup condition (LBB condition) thanks to Poincaré-Friedrichs and solvability of the divergence equation with respect to the norm

$$\|v\|_{\mathbb{V}}^2 = \|(\boldsymbol{v}, q)\|_{\mathbb{V}}^2 = \|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^d)}^2 + \|q\|_{L_2(\Omega)}^2.$$

1. Let $\mathcal{T}_0$ be an initial, conforming triangulation of Ω and let $\mathcal{T}$ be some conforming and shape-regular refinement of $\mathcal{T}_0$:



$\mathcal{T}_0$



$\mathcal{T}$

Conforming Discretization with Adaptive Finite Elements

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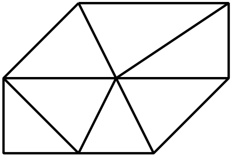

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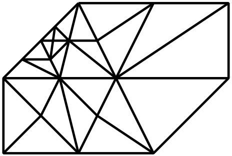

$\mathcal{T}$

2. Let $\mathbb{V}(\mathcal{T}) \subset \mathbb{V}$ be piecewise polynomial finite element space over $\mathcal{T}$ satisfying the single discrete inf-sup condition

$$\inf_{V\in\mathbb{V}(\mathcal{T})} \sup_{W\in\mathbb{V}(\mathcal{T})} \frac{\mathcal{B}[V,W]}{\|V\|_{\mathbb{V}}\|W\|_{\mathbb{V}}} = c(\mathcal{T}) > 0$$

or

$$\inf_{W\in\mathbb{V}(\mathcal{T})} \sup_{V\in\mathbb{V}(\mathcal{T})} \frac{\mathcal{B}[V,W]}{\|V\|_{\mathbb{V}}\|W\|_{\mathbb{V}}} = c(\mathcal{T}) > 0.$$

Remarks on the Discrete Inf-Sup Condition

1. The first discrete inf-sup condition implies injectivity of the discrete operator, whence it is also surjective, and thus, the adjoint operator is injective. This is characterized by the second discrete inf-sup.

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2. Since $\mathbb{V}(\mathcal{T}) \subset \mathbb{V}$, the continuous inf-sup condition implies for any $V \in \mathbb{V}(\mathcal{T})$ the existence of a $w \in \mathbb{V}$ such that

$$\frac{\mathcal{B}[V, w]}{\|V\|_{\mathbb{V}}\|w\|_{\mathbb{V}}} \geq c_*.$$

But in general, $w \notin \mathbb{V}(\mathcal{T})$ and thus the continuous inf-sup does not imply the discrete one.

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But in general, $w \notin \mathbb{V}(\mathcal{T})$ and thus the continuous inf-sup does not imply the discrete one.

3. The continuous inf-sup condition implies the discrete one iff there exists a continuous operator $\Pi \colon \mathbb{V} \to \mathbb{V}(\mathcal{T})$ such that

$$\mathcal{B}[V, w] = \mathcal{B}[V, \Pi w] \qquad \forall V \in \mathbb{V}(\mathcal{T}) \text{ and } w \in \mathbb{V}.$$

Furthermore, $c(\mathcal{T}) \geq \underline{c}_*$ is independent of $\mathcal{T}$ iff there exists a $C > 0$ independent of $\mathcal{T}$ s. th.

$$\|\Pi w\|_{\mathbb{V}} \leq C \|w\|_{\mathbb{V}} \qquad \forall w \in \mathbb{V}.$$

Discrete Problem

The discrete problem reads:

$$U \in \mathbb{V}(\mathcal{T}): \qquad \mathcal{B}[U,V] = \langle f, V\rangle \qquad \forall V \in \mathbb{V}(\mathcal{T}).$$

Since $\mathbb{V}(\mathcal{T}) \subset \mathbb{V}$, the bilinear form $\mathcal{B}$ is continuous. The discrete inf-sup condition implies existence and uniqueness of the discrete solution [Nečas].

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Properties of the discrete solution:

1. A priori bound

$$\|U\|_{\mathbb{V}} \leq c(\mathcal{T})^{-1} \|f\|_{\mathbb{V}^*}$$

2. Galerkin orthogonality

$$\mathcal{B}[U_k - u, V] = 0 \qquad \forall V \in \mathbb{V}(\mathcal{T})$$

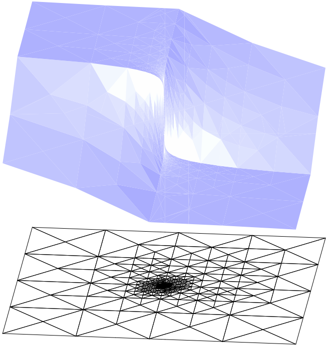
implies the quasi best approximation property [Babuška, '71]

$$\|U - u\|_{\mathbb{V}} \leq \frac{C^*}{c(\mathcal{T})} \inf_{V \in \mathbb{V}(\mathcal{T})} \|V - u\|_{\mathbb{V}}.$$

For coercive forms $\mathcal{B}$ this is Cea's Lemma [Cea '64].

Uniform estimates only for stable discretizations with $c(\mathcal{T}) \geq \underline{c}_* > 0$.

Example of Adaptive Approximation: Singular Solution



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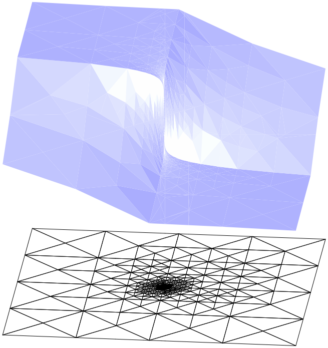

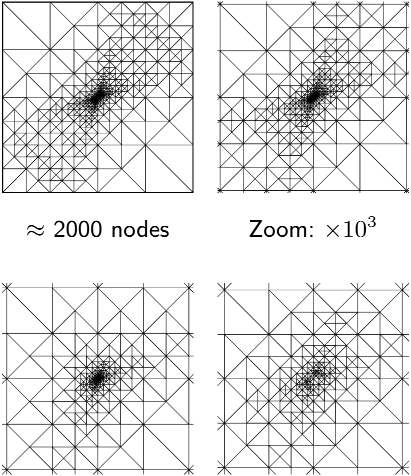

$\approx$ 2000 nodes

Zoom: $\times 10^3$

Zoom: $\times 10^6$

Zoom: $\times 10^9$

The Adaptive Loop

Starting with the initial grid $\mathcal{T}_0$ we use the standard adaptive iteration:

$$\text{SOLVE} \longrightarrow \text{ESTIMATE} \longrightarrow \text{MARK} \longrightarrow \text{REFINE}$$

for computing a sequence $\{\mathcal{T}_k, U_k\}_{k\geq 0}$ of grids and discrete solutions.

- SOLVE: computes the Galerkin approximation $U_k \in \mathbb{V}_k$ to u:
  1. exact integration;
  2. exact numerical algebra;
- ESTIMATE: computes error indicators $\{\mathcal{E}_k(T)\}_{T\in\mathcal{T}_k}$;
- MARK: selects elements in $\mathcal{T}_k$ for refinement;
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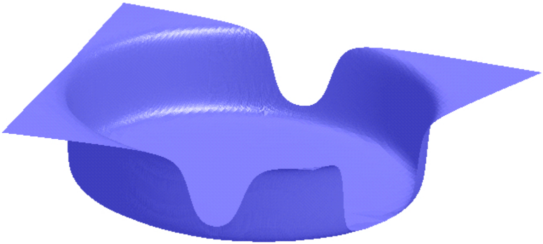
It is not clear, that the discrete solution improves!

Example of Adaptive Approximation: Interior Layer

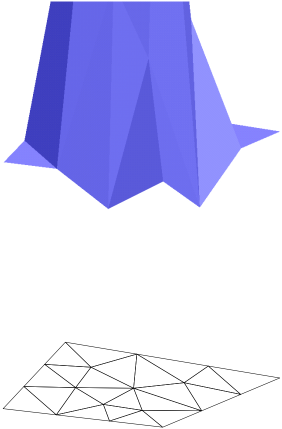
We consider the adaptive approximation to a solution of the Poisson problem with the following features:

- rough ride hand side;
- rough boundary data.

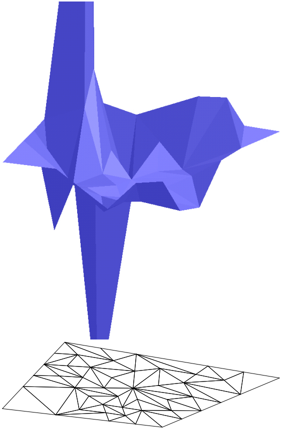
This results in a solution with steep gradients:



Adaptive Iterations 0 and 1

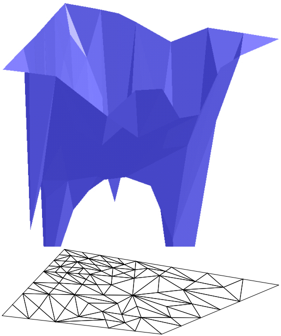


Adaptive iteration 0

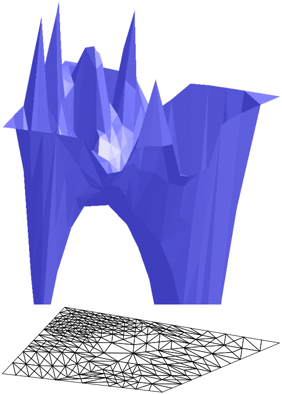


Adaptive iteration 1

Adaptive Iterations 2 and 3

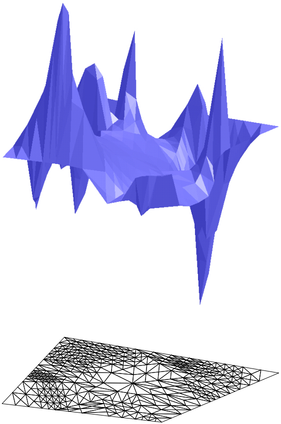


Adaptive iteration 2

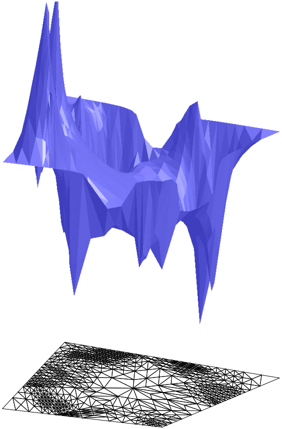


Adaptive iteration 3

Adaptive Iterations 4 and 5

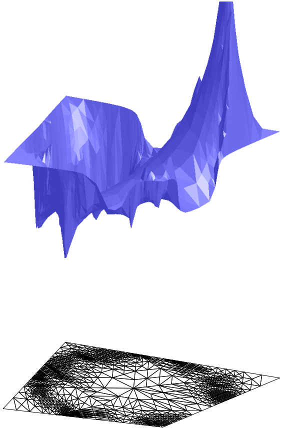


Adaptive iteration 4

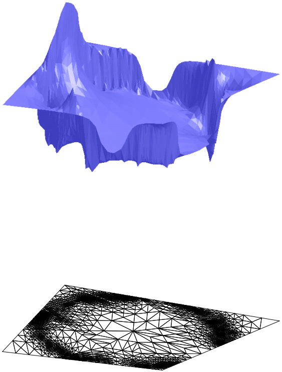


Adaptive iteration 5

Adaptive Iterations 6 and 7

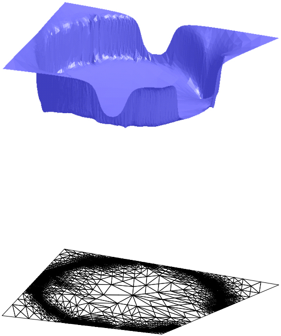


Adaptive iteration 6

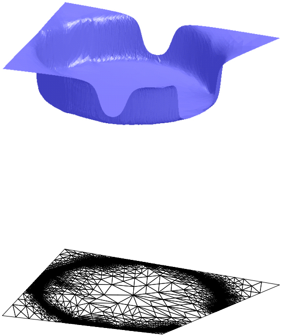


Adaptive iteration 7

Adaptive Iterations 8 and 9

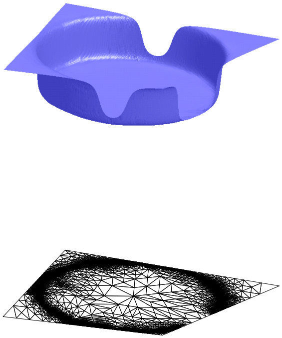


Adaptive iteration 8

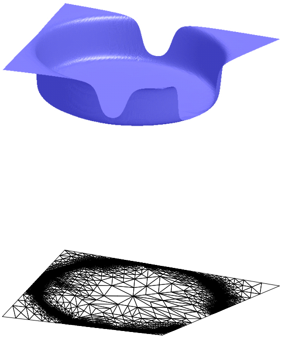


Adaptive iteration 9

Adaptive Iterations 10 and 11



Adaptive iteration 10



Adaptive iteration 11

The Module ESTIMATE

Given $\{\mathcal{T}_k, U_k\}$, compute an estimator for the true error $\|U_k - u\|_{\mathbb{V}}$ in terms of the discrete solution and given data.

A posteriori error estimators are split into local indicators $\mathcal{E}_k(T)$ on elements $T \in \mathcal{T}_k$ and can be summed over subsets $\mathcal{S}_k \subset \mathcal{T}_k$

$$\mathcal{E}_k(\mathcal{S}_k) := \Big(\sum_{T \in \mathcal{S}_k} \mathcal{E}_{\mathcal{T}}^2(T) \Big)^{1/2}.$$

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Properties of the estimator: There exist constants $0 < c_1 \le c_2 < \infty$, solely depending on the shape-regularity of $\mathcal{T}_k$, such that

$$c_1 \|U_k - u\|_{\mathbb{V}}^2 \le \mathcal{E}_k^2(\mathcal{T}_k) \le c_2 \big(\|U_k - u\|_{\mathbb{V}}^2 + \operatorname{osc}_k^2(\mathcal{T}_k) \big).$$

The left inequality is called upper bound the right one lower bound.

- The upper bound only holds globally.
- The lower bound also holds in a local variant:

$$\mathcal{E}_k^2(T) \le c_2 \big(\|U_k - u\|_{\mathbb{V}(\mathcal{U}_k(T))}^2 + \operatorname{osc}_k(\mathcal{U}_k(T))^2 \big)$$

- The oscillation term $\operatorname{osc}_k(\mathcal{T}_k)$ is usually of higher order.

Example: The Residual Estimator and Oscillation

Denote by $h_{\mathcal{T}} \in L_\infty(\Omega)$ the piecewise constant mesh-size function with

$$h_{\mathcal{T}\,|T} = |T|^{1/d} \approx \operatorname{diam}(T), \qquad T \in \mathcal{T}.$$

Poisson problem: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

$$\mathcal{E}^2_{\mathcal{T}}(T) := \|h_{\mathcal{T}}\,(-\Delta U - f)\|^2_{L_2(T)} + \|h^{1/2}_{\mathcal{T}}\,[\![\nabla U]\!]\,\|^2_{L_2(\partial T\cap\Omega)}$$

$$\operatorname{osc}^2_{\mathcal{T}}(T) := \|h_{\mathcal{T}}\,(f_{\mathcal{T}} - f)\|^2_{L_2(T)}.$$

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$$\mathcal{E}_\mathcal{T}^2(T) := \|h_\mathcal{T}\,(-\Delta U - f)\|^2_{L_2(T)} + \|h_\mathcal{T}^{1/2}\,[\![\nabla U]\!]\,\|^2_{L_2(\partial T\cap\Omega)}$$

$$\operatorname{osc}_\mathcal{T}^2(T) := \|h_\mathcal{T}\,(f_\mathcal{T} - f)\|^2_{L_2(T)}.$$

Stokes problem: $-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}$ and $-\operatorname{div}\boldsymbol{u} = 0$ in Ω, $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$

$$\mathcal{E}_\mathcal{T}^2(T) := \|h_\mathcal{T}\,|-\Delta \boldsymbol{U} + \nabla P - \boldsymbol{f}|\,\|^2_{L_2(T)} + \|h_\mathcal{T}^{1/2}\,[\![\nabla \boldsymbol{U}]\!]\,\|^2_{L_2(\partial T\cap\Omega)} + \|\operatorname{div}\boldsymbol{U}\|^2_{L_2(T)}$$

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The Module MARK

Select elements for refinement based on information provided by the indicators $\{\mathcal{E}_k(T)\}_{T\in\mathcal{T}_k}$.

Popular marking strategies are motivated by the equidistribution of the true error on an optimal grid [Babuška, Rheinboldt '78].

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Equidistribution Strategy:

Parameters $\mathsf{TOL} > 0$, $\theta \in [0,1]$

$\mathcal{E}_{\text{limit}} := \theta\, \mathsf{TOL}/\#\mathcal{T}_k^{1/2}$

Maximum Strategy:

Parameter $\nu \in [0,1]$

$\mathcal{E}_{\text{limit}} := \nu \max_{T\in\mathcal{T}_k} \mathcal{E}_k(T)$

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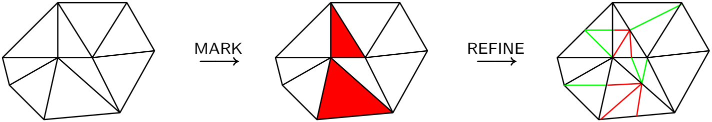
Dörfler Marking invented for the first convergence proof ['96]: Given parameter $\theta \in (0,1]$ select $\mathcal{M}_k \subset \mathcal{T}_k$ such that

$$\theta\mathcal{E}_k(\mathcal{T}_k) \leq \mathcal{E}_k(\mathcal{M}_k).$$

The Module REFINE

Refine all marked elements in $\mathcal{M}_k \subset \mathcal{T}_k$ and create a conforming and shape-regular triangulation $\mathcal{T}_{k+1}$ of Ω.



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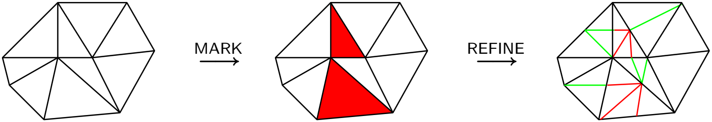

Denote by $\mathbb{T}$ the set of all possible refinements of an initial grid $\mathcal{T}_0$.

Using bisectional refinement yields the following properties:

1. Mesh-size of refined elements is strictly decreased: For the two children T_1, T_2 of any bisected element $T \in \mathcal{T}$ we have $|T_i| = \frac{1}{2}\,|T|$, $i = 1, 2$.
2. Conformity is preserved and shape-regularity of any refinement $\mathcal{T} \in \mathbb{T}$ solely depends on the shape-regularity of $\mathcal{T}_0$.
3. Any sequence $\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_k, \dots$ of generated triangulations is nested which implies nested spaces $\mathbb{V}(\mathcal{T}_0) \subset \mathbb{V}(\mathcal{T}_1) \subset \dots \mathbb{V}(\mathcal{T}_k) \subset \dots \mathbb{V}$ for piecewise polynomials.

Uniform Refinement implies Density

Uniform refinement yields for a sequence $\{\mathcal{T}_k\}_{k\geq 0}$

$$\lim_{k\to\infty} h_{\max}(\mathcal{T}_k) = 0,$$

which implies the following density property of the finite element spaces:

$$\forall v \in \mathbb{V}: \quad \lim_{k\to\infty} \min_{V_k\in\mathbb{V}_k} \|V_k - v\|_{\mathbb{V}} = 0 \qquad \Longrightarrow \qquad \overline{\bigcup_{k\geq 0} \mathbb{V}(\mathcal{T}_k)}^{\|\cdot\|_{\mathbb{V}}} = \mathbb{V}.$$

Proof: Let $\mathbb{W}$ be a dense subspace of $\mathbb{V}$ and let $\Pi_k : \mathbb{W} \to \mathbb{V}(\mathcal{T}_k)$ be an interpolation operator with

$$\|w - \Pi_k w\|_{\mathbb{V}} \lesssim h^q_{\max}(\mathcal{T}_k)\|w\|_{\mathbb{W}},$$

where $q > 0$ depends on $\mathbb{W}$ and $\mathbb{V}(\mathcal{T}_k)$. For instance, $\mathbb{W} = H^2(\Omega)$, $\mathbb{V} = H^1(\Omega)$ and Π_k Lagrange interpolation operator with $q = 1$.
Then for any $v \in \mathbb{V}$ and given $\varepsilon > 0$ first choose $w \in \mathbb{W}$ close to v and then k large enough such that

$$\begin{aligned}\|v - \Pi_k w\|_{\mathbb{V}} &\leq \|v - w\|_{\mathbb{V}} + \|w - \Pi_k w\|_{\mathbb{V}} \\ &\lesssim \|v - w\|_{\mathbb{V}} + h^q_{\max}(\mathcal{T}_k)\|w\|_{\mathbb{V}} \leq \varepsilon.\end{aligned}$$

Convergence: Uniform vs. Adaptive Refinement

For uniform refinement the density of spaces in combination with quasi-best approximation property and stability of the discretization $c(\mathcal{T}) \geq \underline{c}_* > 0$

$$\|U_k - u\|_{\mathbb{V}} \leq \frac{C^*}{\underline{c}_*} \min_{V_k \in \mathbb{V}_k} \|V_k - u\|_{\mathbb{V}} \to 0$$

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Adaptive refinement may not yield this density property, since it may happen that

$$\lim_{k\to\infty} h_{\max}(\mathcal{T}_k) > 0$$

Hence, convergence $U_k \to u$ for $k \to \infty$ is not clear, and can only be true, if u can be approximated by functions $V_k \in \mathbb{V}_k$.

This hinges on properties of the modules

SOLVE, ESTIMATE, MARK, and REFINE.

Outline

Principal idea: "Travel" with the discrete solution and monitor the improvement between two consecutive iterations:

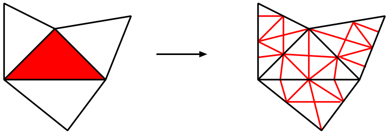
Enforce a strict improvement when going from U_k to U_{k+1}!

MNS algorithm [Morin, Nochetto, S. '00] based on [Dörfler '96]:

- SOLVE: Restriction of problem class: selfadjoint elliptic problems;
- ESTIMATE: Reliable estimator with a discrete local lower bound; needs also oscillation indicators;
- MARK: Dörfler marking for estimator and oscillation:

$$\theta \mathcal{E}_k(\mathcal{T}_k) \le \mathcal{E}_k(\mathcal{M}_k) \qquad \text{and} \qquad \bar\theta \operatorname{osc}_k(\mathcal{T}_k) \le \operatorname{osc}_k(\mathcal{M}_k);$$

- REFINE: Ensure that all marked elements and its direct neighbors are sufficiently refined (interior node property):



Contraction Property of MNS

1. The interior node property gives a discrete lower bound for the error reduction $\|U_k - U_{k+1}\|_{\mathbb{V}}$:

$$\|U_k - U_{k+1}\|_{\mathbb{V}}^2 + \operatorname{osc}_k^2(\mathcal{M}_k) \geq \frac{1}{c_2}\mathcal{E}_k^2(\mathcal{M}_k).$$

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2. Assuming $\operatorname{osc}_k(\mathcal{T}_k) \equiv 0$, Dörfler marking and the upper bound give

$$\|U_k - U_{k+1}\|_{\mathbb{V}}^2 \geq \frac{1}{c_2}\mathcal{E}_k^2(\mathcal{M}_k) \geq \theta^2 \frac{1}{c_2}\mathcal{E}_k^2(\mathcal{T}_k) \geq \theta^2 \frac{c_1}{c_2}\|U_k - u\|_{\mathbb{V}}^2,$$

i. e., the error reduction is a fixed portion of the error.

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i. e., the error reduction is a fixed portion of the error.

3. Restriction to selfadjoint elliptic problems implies orthogonality of the error in the energy norm:

$$\|U_{k+1} - u\|_{\mathbb{V}}^2 = \|U_k - u\|_{\mathbb{V}}^2 - \|U_k - U_{k+1}\|_{\mathbb{V}}^2.$$

which implies the contraction property

$$\|U_{k+1} - u\|_{\mathbb{V}}^2 \leq \underbrace{\left(1 - \theta^2 \frac{c_1}{c_2}\right)}_{<1} \|U_k - u\|_{\mathbb{V}}^2 =: \alpha\|U_k - u\|_{\mathbb{V}}^2.$$

Contraction Result for Selfadjoint Problems

Including marking for oscillation when $\operatorname{osc}_k(\mathcal{T}_k) \not\equiv 0$, one obtains

Theorem (Contraction of Total Error). There exists $\gamma > 0$ and $\alpha < 1$ s. th. MNS achieves

$$\|U_{k+1} - u\|_{\mathbb{V}}^2 + \gamma \operatorname{osc}_{k+1}^2(\mathcal{T}_{k+1}) \le \alpha \left(\|U_k - u\|_{\mathbb{V}}^2 + \gamma \operatorname{osc}_k^2(\mathcal{T}_k)\right).$$

[Mekchay, Nochetto '05] based on [Chen, Feng '04] and [Morin, Nochetto, S. '00,'02,'03].

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Extensions to other linear and nonlinear problems:

- Bänsch, Morin & Nochetto; Carstensen & Hoppe; Cascon, Nochetto & S., Chen, Holst & Xu, Becker & al. . .
- Veeser; S. & Veeser; Carstensen; Diening & Kreuzer.

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Recent result: The standard AFEM without interior node property and without marking for oscillation yields for a $\gamma > 0$ and $0 < \alpha < 1$

$$\|U_{k+1} - u\|_{\mathbb{V}}^2 + \gamma \mathcal{E}_{k+1}^2(\mathcal{T}_{k+1}) \le \alpha \left(\|U_k - u\|_{\mathbb{V}}^2 + \gamma \mathcal{E}_k^2(\mathcal{T}_k)\right).$$

[Cascon, Kreuzer, Nochetto, & S. '08]

Decay Rate for Selfadjoint Elliptic Problems in Terms of DOFs

For adaptive methods, the speed of convergence has to be measured in terms of Degrees Of Freedom (DOFs):

$$\|U_k - u\|_{\mathbb{V}} \lesssim (\#\mathcal{T}_k - \#\mathcal{T}_0)^{-s} \quad \text{instead of} \quad \|U_k - u\|_{\mathbb{V}} \lesssim h_{\max}^q(\mathcal{T}_k).$$

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Based on a contraction property of AFEM for some suitable error notion, for instance the total error for MNS, one can proof the following result:

Theorem (Optimality). The sequence of Ritz-Galerkin solutions $\{U_k\}_{k\ge 0}$ is quasi-optimal with respect to DOFs, i. e.,

$$\min_{\mathcal{T}\in\mathbb{T}_N} \min_{V\in\mathbb{V}(\mathcal{T})} \|V-u\|_{\mathbb{V}} \lesssim N^{-s} \quad \Longrightarrow \quad \|U_k - u\|_{\mathbb{V}} \lesssim (\#\mathcal{T}_k - \#\mathcal{T}_0)^{-s},$$

where $\mathbb{T}_N = \{\mathcal{T}\in\mathbb{T} \mid \#\mathcal{T} - \#\mathcal{T}_0 \le N\}$ (plus decay of oscillation).

- Binev, Dahmen, DeVore '04, Stevenson '05: Modification of MNS with additional coarsening;
- Stevenson '05: Nearly standard AFEM with an inner loop to reduce oscillation;
- Cascon, Kreuzer, S., Nochetto '08: Standard AFEM.

A Lot of Open Questions

The above result cannot be (directly) generalized to problems that are not related to an energy minimization:

- Diffusion-advection problems
- Saddle point problems
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For some nonlinear problems MNS does not even yield a contraction.

But: AFEM is working well for these problems.

Outline

Convergence of AFEM: Observe Progress

Principal idea: Observe the full sequence $\{U_k\}_{k\geq 0}$ of discrete solutions as they pass by:

Determine properties of the modules of AFEM which guarantee convergence.

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Module SOLVE:

1. Conforming and nested approximation:

$$\forall \mathcal{T} \in \mathbb{T}: \quad \mathbb{V}(\mathcal{T}) \subset \mathbb{V} \qquad \text{and} \qquad \forall \mathcal{T} \leq \mathcal{T}_* \in \mathbb{T}: \quad \mathbb{V}(\mathcal{T}) \subset \mathbb{V}(\mathcal{T}_*).$$

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2. Stable discretization of (P):

$$\forall \mathcal{T} \in \mathbb{T}: \qquad \inf_{V \in \mathbb{V}(\mathcal{T})} \sup_{W \in \mathbb{V}(\mathcal{T})} \frac{\mathcal{B}[V,W]}{\|V\|_{\mathbb{V}} \|W\|_{\mathbb{V}}} \geq \underline{c}_*$$

with a fixed constant $\underline{c}_* > 0$ solely depending on the bilinear form $\mathcal{B}$ and $\mathbb{T}$, but not on a particular $\mathcal{T} \in \mathbb{T}$.

Convergence of Galerkin Solutions [Morin, S., Veeser '08]

Lemma (Convergence of Galerkin Solutions). The assumptions on SOLVE imply the existence of $u_\infty \in \mathbb{V}$ such that

$$\lim_{k\to\infty} \|U_k - u_\infty\|_{\mathbb{V}} = 0,$$

and u_∞ is the solution of

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3. Quasi-best approximation property of U_k with respect to u_∞ yields

$$\|U_k - u_\infty\|_{\mathbb{V}} \leq \frac{C^*}{\underline{c}_*} \inf_{V_k \in \mathbb{V}(\mathcal{T})} \|V_k - u_\infty\|_{\mathbb{V}} \to 0$$

by construction of $\mathbb{V}_\infty$.

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Lemma (Convergence of Mesh Size Functions). The sequence of mesh-size functions $\{h_k\}_{k\geq 0} \subset L_\infty(\Omega)$ defined as

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converges uniformly to some $h_\infty \in L_\infty(\Omega)$, i. e.,

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2. The basic property of refinement by bisection implies: for any $x \in \Omega$ there holds

$$\text{either} \qquad h_{k+1}(x) = h_k(x) \qquad \text{or} \qquad h_{k+1}(x) \leq 2^{-1/d} h_k(x).$$

This can be utilized to conclude uniform convergence.

Limits and Splitting of the Grid

Observations:

1. If it happens that $\mathbb{V}_\infty = \mathbb{V}$, then $u_\infty = u$ and we have convergence, i. e.,

$$\lim_{k\to\infty} \|U_k - u\|_{\mathbb{V}} = 0.$$

2. $\mathbb{V}_\infty \neq \mathbb{V}$ is equivalent to $h_\infty \not\equiv 0$ in Ω, i. e., $h_\infty(x) > 0$ for some $x \in \Omega$.
3. $h_\infty(x) > 0$ implies, that there exists $K = K(x)$ and $T \in \mathcal{T}_K$ such that $x \in T$ and $T \in \mathcal{T}_k$ for all $k \geq K$.

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This motivates the splitting of $\mathcal{T}_k$:

$$\mathcal{T}_k^+ := \bigcap_{\ell\geq k} \mathcal{T}_\ell \qquad \text{and} \qquad \mathcal{T}_k^0 := \mathcal{T}_k \setminus \mathcal{T}_k^+.$$

elements that are no longer refined, and elements that are refined at least once.

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3. $h_\infty(x) > 0$ implies, that there exists $K = K(x)$ and $T \in \mathcal{T}_K$ such that $x \in T$ and $T \in \mathcal{T}_k$ for all $k \geq K$.

This motivates the splitting of $\mathcal{T}_k$:

$$\mathcal{T}_k^+ := \bigcap_{\ell \geq k} \mathcal{T}_\ell \qquad \text{and} \qquad \mathcal{T}_k^0 := \mathcal{T}_k \setminus \mathcal{T}_k^+.$$

elements that are no longer refined, and elements that are refined at least once.

Corollary. The splitting of $\mathcal{T}_k$ implies for the mesh size functions in $\Omega(\mathcal{T}_k^0)$:

$$\lim_{k\to\infty} \|h_k\|_{\infty;\Omega(\mathcal{T}_k^0)} = 0.$$

Additional Assumption on Module SOLVE

Local approximability: Let $\mathbb{W} \subset \mathbb{V}$ be a dense sub-space and let $\Pi_k \in L(\mathbb{W}, \mathbb{V}_k)$ be a continuous, linear interpolation operator with

$$\forall w \in \mathbb{W},\ \forall T \in \mathcal{T}_k: \qquad \|w - \Pi_k w\|_{\mathbb{V}(T)} \lesssim \|h_k^q\|_{\infty;T} \|w\|_{\mathbb{W}(T)},$$

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Lemma (Local Density). Convergence of the mesh size function in $\Omega(\mathcal{T}_k^0)$ implies local density of the finite element spaces

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Proof: Local density follows from local approximability:

$$\begin{aligned} \|v - \Pi_k w\|_{\mathbb{V}(\Omega(\mathcal{T}_k^0))} &\le \|v - w\|_{\mathbb{V}(\Omega(\mathcal{T}_k^0))} + \|w - \Pi_k w\|_{\mathbb{V}(\Omega(\mathcal{T}_k^0))} \\ &\le \|v - w\|_{\mathbb{V}(\Omega)} + C\,\|h_k^q\|_{\infty;\Omega(\mathcal{T}_k^0)} \|w\|_{\mathbb{W}(\Omega)}. \end{aligned}$$

Now, choose first $w \in \mathbb{W}$ close to v and then k large to make the right hand side small.

Observation: Obviously, ESTIMATE has to control the locally induced error in $\Omega(\mathcal{T}_k^+)$ and MARK $\mathcal{E}_k(\mathcal{T}_k^+)$.

Assumptions on Modules ESTIMATE, MARK, and REFINE

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$$\forall v \in \mathbb{V}: \qquad |\langle \mathcal{R}(U_k),\, v\rangle| \lesssim \sum_{T \in \mathcal{T}_k} \mathcal{E}_k(T) \|v\|_{\mathbb{V}(\mathcal{U}_k(T))}.$$

  Stability of the indicators

$$\forall T \in \mathcal{T}_k: \qquad \mathcal{E}_k(T) \lesssim \|U_k\|_{\mathbb{V}(\mathcal{U}_k(T))} + \|D\|_{2;\mathcal{U}_k(T)}$$

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$$\forall T \in \mathcal{T}_k \setminus \mathcal{M}_k: \qquad \mathcal{E}_k(T) \le g(\max\{\mathcal{E}_k(T) \mid T \in \mathcal{M}_k\}),$$

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Consequences

Lemma. The estimator $\mathcal{E}_k(\mathcal{T}_k)$ is uniformly bounded, i. e.,

$$\mathcal{E}_k(\mathcal{T}_k) \le \Lambda$$

and the maximal indicator vanishes in the limit:

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Steps of the Proof: (with $\mathcal{U}_k(T)$ replaced by T, ...)

1. Stability of the discretization and the indicators yields

$$\mathcal{E}_k^2(\mathcal{T}_k) \lesssim \sum_{T\in\mathcal{T}_k} \|U_k\|_{\mathbb{V}(T)}^2 + \|D\|_{2;T}^2 \lesssim \|U_k\|_{\mathbb{V}(\Omega)}^2 + \|D\|_{2;\Omega}^2 \le \Lambda$$

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2. Let $T_k \in \mathcal{M}_k$ s. th. $\mathcal{E}_k(T_k) = \max\{\mathcal{E}_k(T) \mid T \in \mathcal{M}_k\}$. Since $T_k \in \mathcal{M}_k \subset \mathcal{T}_k^0$ convergence of the mesh size function gives

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by convergence of Galerkin solutions and continuity of norms with respect to the Lebesgue measure. Now, assumption on marking yields the claim.

Convergence Without Lower Bound [S. '??]

Theorem (Convergence of AFEM). The standard AFEM with a reliable estimator achieves

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Sketch of the Proof: We already know the strong convergence $U_k \to u_\infty$ in $\mathbb{V}$, and thus it remains to show $u_\infty = u$, for instance by proving

$$\mathcal{R}(u_\infty) = 0 \quad \text{in } \mathbb{V}^*.$$

Since $\mathbb{W}$ is dense in $\mathbb{V}$ it is sufficient to prove

$$\lim_{k\to\infty} \langle \mathcal{R}(U_k),\, w\rangle = \langle \mathcal{R}(u_\infty),\, w\rangle = 0 \qquad \forall w \in \mathbb{W}.$$

Convergence Without Lower Bound

For $k \geq \ell$ it holds $\mathcal{T}_\ell^+ \subset \mathcal{T}_k^+ \subset \mathcal{T}_k$ and $\Omega(\mathcal{T}_\ell^0) = \Omega(\mathcal{T}_k \setminus \mathcal{T}_\ell^+)$.

Galerkin orthogonality in combination with the upper bound gives for any $w \in \mathbb{W}$ with $\|w\|_{\mathbb{W}} = 1$

$$|\langle \mathcal{R}(U_k),\, w \rangle| = |\langle \mathcal{R}(U_k),\, w - \Pi_k w \rangle| \lesssim \sum_{T \in \mathcal{T}_k} \mathcal{E}_k(T) \|w - \Pi_k w\|_{\mathbb{V}(T)}$$

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$$\lesssim \sum_{T\in\mathcal{T}_k\setminus\mathcal{T}_\ell^+} \mathcal{E}_k(T)\|w - \Pi_k w\|_{\mathbb{V}(T)} + \sum_{T\in\mathcal{T}_\ell^+} \mathcal{E}_k(T)\|w - \Pi_k w\|_{\mathbb{V}(T)}$$

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&\lesssim \mathcal{E}_k(\mathcal{T}_k) \|w - \Pi_k w\|_{\mathbb{V}(\Omega(\mathcal{T}_\ell^0))} + \mathcal{E}_k(\mathcal{T}_\ell^+) \|w - \Pi_k w\|_{\mathbb{V}(\Omega)}
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&\lesssim \Lambda \|h_k^q\|_{\infty;\Omega(\mathcal{T}_\ell^0)} + \mathcal{E}_k(\mathcal{T}_\ell^+) \\
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Let $\varepsilon > 0$ be arbitrary. Convergence of mesh size functions allows to first choose ℓ s. th.

$$\Lambda \| h_\ell^q \|_{\infty;\Omega(\mathcal{T}_\ell^0)} \le \frac{\varepsilon}{2}.$$

Convergence of the maximal indicator then allows to choose $k \ge \ell$ s. th.

$$\mathcal{E}_k(T) \le \frac{\varepsilon}{2} (\#\mathcal{T}_\ell^+)^{-1/2} \quad \forall T \in \mathcal{T}_\ell^+ \qquad \Longrightarrow \qquad \mathcal{E}_k(\mathcal{T}_\ell^+) \le \frac{\varepsilon}{2}.$$

In summary $|\langle \mathcal{R}(U_k),\, w \rangle| \lesssim \varepsilon$ for k sufficiently large, which implies $\langle \mathcal{R}(U_k),\, w \rangle \to 0$ as $k \to \infty$.

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Remark: The assumption on marking can be weakened such that it becomes essentially necessary:

$$\lim_{k \to \infty} \max\{ \mathcal{E}_k(T) \,|T \in \mathcal{M}_k \} = 0$$

$$\Longrightarrow \qquad \forall T \in \bigcup_{\ell \geq 0} \mathcal{T}_\ell^+ : \qquad \lim_{k \to \infty} \mathcal{E}_k(T) = 0.$$

Convergence of the Estimator

This result holds true for non-efficient estimators, even in the case

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Sketch of the Proof: Split

$$\begin{aligned}\mathcal{E}_k^2(\mathcal{T}_k) &= \mathcal{E}_k^2(\mathcal{T}_k \setminus \mathcal{T}_\ell^+) + \mathcal{E}_k^2(\mathcal{T}_\ell^+)\\ &\lesssim \|U_k - u\|_{\mathbb{V}(\Omega(\mathcal{T}_k\setminus\mathcal{T}_\ell^+))} + \operatorname{osc}_k(\Omega(\mathcal{T}_k \setminus \mathcal{T}_\ell^+)) + \mathcal{E}_k^2(\mathcal{T}_\ell^+) \le \varepsilon\end{aligned}$$

by first choosing ℓ and then $k \ge \ell$ sufficiently large.

Outline

Concluding Remarks

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   - the adaptive method must not overlook possible error sources;
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Thank you for your attention!