Muckenhoupt-Wheeden conjecture for commutators

I.P. Rivera-Ríos
based on a joint work with C. Pérez

18 de Marzo 2016 - IMAL Santa Fe

Outline

Outline

Calderón-Zygmund operators

The model example is the Hilbert transform:

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{1}{x-y} f(y)dy.$$

It's bounded on L^p

$$\|Hf\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R})}$$

and of weak type $(1,1)$

$$|\{x \in \mathbb{R} : |Hf(x)| > t\}| \leq \frac{C}{t} \int_{\mathbb{R}} |f(x)|dx.$$

Calderón-Zygmund operators

The model example is the Hilbert transform:

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{1}{x-y} f(y)dy.$$

It's bounded on L^p

$$\|Hf\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R})}$$

and of weak type $(1,1)$

$$|\{x \in \mathbb{R} : |Hf(x)| > t\}| \leq \frac{C}{t} \int_{\mathbb{R}} |f(x)|dx.$$

Calderón-Zygmund operators

The model example is the Hilbert transform:

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{1}{x-y} f(y)dy.$$

It's bounded on L^p

$$\|Hf\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R})}$$

and of weak type $(1,1)$

$$|\{x \in \mathbb{R} : |Hf(x)| > t\}| \leq \frac{C}{t} \int_{\mathbb{R}} |f(x)|dx.$$

Definition

A Calderón-Zygmund operator T (CZO) is an operator bounded on $L^2(\mathbb{R}^n)$ that admits the following representation

$$Tf(x) = \int K(x,y)f(y)dy$$

with $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $x \notin \operatorname{supp} f$ and where $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\} \longrightarrow \mathbb{R}$ has the following properties

Size condition: $|K(x,y)| \leq C_2 \frac{1}{|x-y|^n} \qquad x \neq 0.$

Smoothness condition (Hölder-Lipschitz):

$$|K(x,y) - K(x,z)| \leq C_1 \frac{|y-z|^\delta}{|x-y|^{n+\delta}} \qquad \frac{1}{2}|x-y| > |y-z|$$

$$|K(x,y) - K(z,y)| \leq C_1 \frac{|x-z|^\delta}{|x-y|^{n+\delta}} \qquad \frac{1}{2}|x-y| > |x-z|$$

where $C_1 > 0$ and $C_2 > 0$ are constants independent of x, y, z.

The same boundedness results for Hilbert transform also hold for CZO.

Definition

A Calderón-Zygmund operator T (CZO) is an operator bounded on $L^2(\mathbb{R}^n)$ that admits the following representation

$$Tf(x) = \int K(x,y)f(y)dy$$

with $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $x \notin \operatorname{supp} f$ and where $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\} \longrightarrow \mathbb{R}$ has the following properties

Size condition: $|K(x,y)| \leq C_2 \frac{1}{|x-y|^n} \qquad x \neq 0.$

Smoothness condition (Hölder-Lipschitz):

$$|K(x,y) - K(x,z)| \leq C_1 \frac{|y-z|^\delta}{|x-y|^{n+\delta}} \qquad \frac{1}{2}|x-y| > |y-z|$$

$$|K(x,y) - K(z,y)| \leq C_1 \frac{|x-z|^\delta}{|x-y|^{n+\delta}} \qquad \frac{1}{2}|x-y| > |x-z|$$

where $C_1 > 0$ and $C_2 > 0$ are constants independent of x, y, z.

The same boundedness results for Hilbert transform also hold for CZO.

Outline

BMO and commutators

Definition

We say a locally integrable function b has bounded mean oscillation, $b \in BMO$ if

$$\sup_Q \frac{1}{|Q|}\int_Q |b(x)-b_Q|dx < \infty \qquad \text{where} \qquad b_Q = \frac{1}{|Q|}\int_Q b(x)dx$$

Theorem (John-Nirenberg)

There exist two positive constants $\lambda > 0$ and $C > 0$ such that for any $b \in BMO$, $\sup_Q \frac{1}{|Q|}\int_Q \exp\left(\frac{\lambda}{\|b\|_{BMO}}|b(x)-b_Q|\right)dx \leq C$

Definition

Let T a CZO, $b \in BMO$. We define the commutator $[b,T]$ as

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x).$$

BMO and commutators

Definition

We say a locally integrable function b has bounded mean oscillation, $b \in BMO$ if

$$\sup_Q \frac{1}{|Q|}\int_Q |b(x) - b_Q|dx < \infty \qquad \text{where} \qquad b_Q = \frac{1}{|Q|}\int_Q b(x)dx$$

Theorem (John-Nirenberg)

There exist two positive constants $\lambda > 0$ and $C > 0$ such that for any $b \in BMO$, $\sup_Q \frac{1}{|Q|}\int_Q \exp\left(\frac{\lambda}{\|b\|_{BMO}}|b(x) - b_Q|\right) dx \leq C$

Definition

Let T a CZO, $b \in BMO$. We define the commutator $[b, T]$ as

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

BMO and commutators

Definition

We say a locally integrable function b has bounded mean oscillation, $b \in BMO$ if

$$\sup_Q \frac{1}{|Q|}\int_Q |b(x)-b_Q|dx < \infty \qquad \text{where} \qquad b_Q = \frac{1}{|Q|}\int_Q b(x)dx$$

Theorem (John-Nirenberg)

There exist two positive constants $\lambda > 0$ and $C > 0$ such that for any $b \in BMO$, $\sup_Q \frac{1}{|Q|}\int_Q \exp\left(\frac{\lambda}{\|b\|_{BMO}}|b(x)-b_Q|\right)dx \le C$

Definition

Let T a CZO, $b \in BMO$. We define the commutator $[b,T]$ as

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x).$$

Boundedness of commutators

Theorem (Coifman,Rochberg,Weiss, [CRW])

Let T a CZO and $b \in BMO$ then $[b, T]$ is bounded on L^p

Remark

The operator $[b, T]$ is not of weak type $(1, 1)$.

We have the following substitute:

Theorem (Pérez [CP1])

Let T a CZO and $b \in BMO$ then

$$|\{x \in \mathbb{R}^n : |[b, T]f(x)| > \lambda\}| \leq c \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO} \frac{|f(x)|}{\lambda}\right) dx$$

where $\Phi(t) = t\left(1 + \log^+ t\right)$.

Boundedness of commutators

Theorem (Coifman,Rochberg,Weiss, [CRW])

Let T a CZO and $b \in BMO$ then $[b, T]$ is bounded on L^p

Remark

The operator $[b, T]$ is not of weak type $(1, 1)$.

We have the following substitute:

Theorem (Pérez [CP1])

Let T a CZO and $b \in BMO$ then

$$|\{x \in \mathbb{R}^n : |[b, T]f(x)| > \lambda\}| \leq c \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO} \frac{|f(x)|}{\lambda}\right) dx$$

where $\Phi(t) = t\left(1 + \log^+ t\right)$.

Boundedness of commutators

Theorem (Coifman,Rochberg,Weiss, [CRW])

Let T a CZO and $b \in BMO$ then $[b, T]$ is bounded on L^p

Remark

The operator $[b, T]$ is not of weak type $(1, 1)$.

We have the following substitute:

Theorem (Pérez [CP1])

Let T a CZO and $b \in BMO$ then

$$|\{x \in \mathbb{R}^n : |[b, T]f(x)| > \lambda\}| \leq c \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO} \frac{|f(x)|}{\lambda}\right) dx$$

where $\Phi(t) = t\left(1 + \log^+ t\right)$.

Outline

Weights

Definition

We say w is a weight if it is a non-negative locally integrable function.

Definition (A_p class)

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\right)^{\frac{1}{p-1}} < \infty \quad p > 1$$
$$Mw(x) \leq \kappa w(x) \quad \text{a.e.} \qquad p = 1$$

We define $[w]_{A_1} = \inf\{\kappa > 0 : Mw(x) \leq \kappa w(x) \quad \text{a.e.}\}$.

Weights

Definition

We say w is a weight if it is a non-negative locally integrable function.

Definition (A_p class)

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\right)^{\frac{1}{p-1}} < \infty \quad p > 1$$
$$Mw(x) \le \kappa w(x) \quad \text{a.e.} \qquad p = 1$$

We define $[w]_{A_1} = \inf\{\kappa > 0 : Mw(x) \le \kappa w(x) \quad \text{a.e.}\}$.

Weights

Properties

- If $1 < p < \infty$, $w \in A_p$ if and only if $M : L^p(w) \longrightarrow L^p(w)$. Furthermore $\|M\|_{L^p(w)} \lesssim [w]_{A_p}^{\frac{1}{p-1}}$
- If $p = 1$, $w \in A_1$ if and only if $M : L^1(w) \longrightarrow L^{1,\infty}(w)$.
- The A_p classes are increasing

$$p \leq q \Rightarrow A_p \subseteq A_q$$

Definition

$A_\infty = \bigcup_{p \geq 1} A_p$

It can be proved that $w \in A_\infty \iff [w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(\chi_Q w)(x)dx < \infty$

Weights

Properties

- If $1 < p < \infty$, $w \in A_p$ if and only if $M : L^p(w) \longrightarrow L^p(w)$. Furthermore $\|M\|_{L^p(w)} \lesssim [w]_{A_p}^{\frac{1}{p-1}}$
- If $p = 1$, $w \in A_1$ if and only if $M : L^1(w) \longrightarrow L^{1,\infty}(w)$.
- The A_p classes are increasing

$$p \leq q \Rightarrow A_p \subseteq A_q$$

Definition

$A_\infty = \bigcup_{p\geq 1} A_p$

It can be proved that $w \in A_\infty \iff [w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(\chi_Q w)(x)dx < \infty$

Weights

Properties

- If $1 < p < \infty$, $w \in A_p$ if and only if $M : L^p(w) \longrightarrow L^p(w)$. Furthermore $\|M\|_{L^p(w)} \lesssim [w]_{A_p}^{\frac{1}{p-1}}$
- If $p = 1$, $w \in A_1$ if and only if $M : L^1(w) \longrightarrow L^{1,\infty}(w)$.
- The A_p classes are increasing

$$p \leq q \Rightarrow A_p \subseteq A_q$$

Definition

$A_\infty = \bigcup_{p\geq 1} A_p$

It can be proved that $w \in A_\infty \iff [w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(\chi_Q w)(x)dx < \infty$

Weights

Properties

- If $1 < p < \infty$, $w \in A_p$ if and only if $M : L^p(w) \longrightarrow L^p(w)$. Furthermore $\|M\|_{L^p(w)} \lesssim [w]_{A_p}^{\frac{1}{p-1}}$
- If $p = 1$, $w \in A_1$ if and only if $M : L^1(w) \longrightarrow L^{1,\infty}(w)$.
- The A_p classes are increasing

$$p \leq q \Rightarrow A_p \subseteq A_q$$

Definition

$A_\infty = \bigcup_{p \geq 1} A_p$

It can be proved that $w \in A_\infty \iff [w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(\chi_Q w)(x)dx < \infty$

Weights

Properties

- If $1 < p < \infty$, $w \in A_p$ if and only if $M : L^p(w) \longrightarrow L^p(w)$. Furthermore $\|M\|_{L^p(w)} \lesssim [w]_{A_p}^{\frac{1}{p-1}}$
- If $p = 1$, $w \in A_1$ if and only if $M : L^1(w) \longrightarrow L^{1,\infty}(w)$.
- The A_p classes are increasing

$$p \leq q \Rightarrow A_p \subseteq A_q$$

Definition

$A_\infty = \bigcup_{p\geq 1} A_p$

It can be proved that $w \in A_\infty \iff [w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(\chi_Q w)(x)dx < \infty$

Theorem

Let T be a CZO then we have that

- *If $w \in A_p$ for $1 < p < \infty$ T is bounded on $L^p(w)$ and* $\|T\|_{L^p(w)} \lesssim [w]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}}$
- *If $w \in A_1$ T is of weak type $(1,1)$ and* $\|T\|_{L^1(w)\to L^{1,\infty}(w)} \lesssim [w]_{A_1}\log(e+[w]_{A_1})$

Theorem (Coifman, Rochberg, Weiss - Pérez [CRW, CP2])

Let T be a CZO and $b \in BMO$. Then

- *If $w \in A_p$ $1 < p < \infty$, $[b,T]$ is bounded on $L^p(w)$ and*

  $\|[b,T]\|_{L^p(w)} \leq [w]_{A_p}^{2\max\left\{1,\frac{1}{p-1}\right\}}$
- *If $w \in A_1$ then*

$$w\left(\{x \in \mathbb{R}^n : |[b,T]f(x)| > \lambda\}\right) \leq c_w \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{\lambda}\right) w(x)dx$$

where $\Phi(t) = t\left(1+\log^+ t\right)$.

Theorem

Let T be a CZO then we have that

- *If $w \in A_p$ for $1 < p < \infty$ T is bounded on $L^p(w)$ and* $\|T\|_{L^p(w)} \lesssim [w]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}}$
- *If $w \in A_1$ T is of weak type $(1,1)$ and* $\|T\|_{L^1(w)\to L^{1,\infty}(w)} \lesssim [w]_{A_1}\log(e+[w]_{A_1})$

Theorem (Coifman,Rochberg,Weiss - Pérez [CRW, CP2])

Let T be a CZO and $b \in BMO$. Then

- *If $w \in A_p$ $1 < p < \infty$, $[b,T]$ is bounded on $L^p(w)$ and*
  $\|[b,T]\|_{L^p(w)} \leq [w]_{A_p}^{2\max\left\{1,\frac{1}{p-1}\right\}}$
- *If $w \in A_1$ then*

$$w\left(\{x \in \mathbb{R}^n : |[b,T]f(x)| > \lambda\}\right) \leq c_w \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{\lambda}\right) w(x)dx$$

where $\Phi(t) = t\left(1+\log^+ t\right)$.

Theorem

Let T be a CZO then we have that

- *If $w \in A_p$ for $1 < p < \infty$ T is bounded on $L^p(w)$ and* $\|T\|_{L^p(w)} \lesssim [w]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}}$
- *If $w \in A_1$ T is of weak type $(1,1)$ and* $\|T\|_{L^1(w)\to L^{1,\infty}(w)} \lesssim [w]_{A_1}\log(e+[w]_{A_1})$

Theorem (Coifman,Rochberg,Weiss - Pérez [CRW, CP2])

Let T be a CZO and $b \in BMO$. Then

- *If $w \in A_p$ $1 < p < \infty$, $[b,T]$ is bounded on $L^p(w)$ and*

  $\|[b,T]\|_{L^p(w)} \leq [w]_{A_p}^{2\max\left\{1,\frac{1}{p-1}\right\}}$
- *If $w \in A_1$ then*

$$w\left(\{x\in\mathbb{R}^n : |[b,T]f(x)|>\lambda\}\right) \leq c_w \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{\lambda}\right) w(x)dx$$

  where $\Phi(t) = t\left(1+\log^+ t\right)$.

Theorem

Let T be a CZO then we have that

- *If* $w \in A_p$ *for* $1 < p < \infty$ T *is bounded on* $L^p(w)$ *and* $\|T\|_{L^p(w)} \lesssim [w]_{A_p}^{\max\left\{1, \frac{1}{p-1}\right\}}$
- *If* $w \in A_1$ T *is of weak type* $(1,1)$ *and* $\|T\|_{L^1(w) \to L^{1,\infty}(w)} \lesssim [w]_{A_1} \log(e + [w]_{A_1})$

Theorem (Coifman,Rochberg,Weiss - Pérez [CRW, CP2])

Let T be a CZO and $b \in BMO$. Then

- *If* $w \in A_p$ $1 < p < \infty$, $[b, T]$ *is bounded on* $L^p(w)$ *and* $\|[b, T]\|_{L^p(w)} \leq [w]_{A_p}^{2\max\left\{1, \frac{1}{p-1}\right\}}$
- *If* $w \in A_1$ *then*

$$w\left(\{x \in \mathbb{R}^n : |[b, T]f(x)| > \lambda\}\right) \leq c_w \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO} \frac{|f(x)|}{\lambda}\right) w(x)dx$$

where $\Phi(t) = t\left(1 + \log^+ t\right)$.

Outline

Orlicz maximal functions

The Hardy-Littlewood maximal function is defined as

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(x)| dx$$

where each Q is a cube with its sides parallel to the axis.
If we replace the standard averages for "more general ones" we can define more maximal operators.

Orlicz maximal functions

The Hardy-Littlewood maximal function is defined as

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(x)|dx$$

where each Q is a cube with its sides parallel to the axis.
If we replace the standard averages for "more general ones" we can define more maximal operators.

Orlicz averages

Definition

Let $\Phi : [0,\infty) \longrightarrow (0,\infty)$ be a Young function, i.e. a convex, increasing function such that $\Phi(0) = 0$,

$$\|f\|_{\Phi,Q} = \inf\left\{\lambda > 0\,;\, \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \leq 1\right\}.$$

Clearly $\Phi(t) = t^p$

$$\|f\|_{\Phi,Q} = \left(\frac{1}{|Q|}\int_Q |f(x)|^p dx\right)^{\frac{1}{p}}.$$

Another basic example is given by the function $\Psi(t) = \exp(t) - 1$. By John-Nirenberg's theorem if $f \in BMO$, we have that

$$\|f\|_{\exp(L),Q} = \|f\|_{\Psi(L),Q} \leq c\|f\|_{BMO}$$

Orlicz averages

Definition

Let $\Phi : [0,\infty) \longrightarrow (0,\infty)$ be a Young function, i.e. a convex, increasing function such that $\Phi(0) = 0$,

$$\|f\|_{\Phi,Q} = \inf\left\{\lambda > 0\,;\, \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1\right\}.$$

Clearly $\Phi(t) = t^p$

$$\|f\|_{\Phi,Q} = \left(\frac{1}{|Q|}\int_Q |f(x)|^p dx\right)^{\frac{1}{p}}.$$

Another basic example is given by the function $\Psi(t) = \exp(t) - 1$. By John-Nirenberg's theorem if $f \in BMO$, we have that

$$\|f\|_{\exp(L),Q} = \|f\|_{\Psi(L),Q} \leq c\|f\|_{BMO}$$

Orlicz averages

Definition

Let $\Phi : [0,\infty) \longrightarrow (0,\infty)$ be a Young function, i.e. a convex, increasing function such that $\Phi(0) = 0$,

$$\|f\|_{\Phi,Q} = \inf\left\{\lambda > 0\,;\, \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1\right\}.$$

Clearly $\Phi(t) = t^p$

$$\|f\|_{\Phi,Q} = \left(\frac{1}{|Q|}\int_Q |f(x)|^p dx\right)^{\frac{1}{p}}.$$

Another basic example is given by the function $\Psi(t) = \exp(t) - 1$. By John-Nirenberg's theorem if $f \in BMO$, we have that

$$\|f\|_{\exp(L),Q} = \|f\|_{\Psi(L),Q} \leq c\|f\|_{BMO}$$

Orlicz maximal functions

Definition

We define the maximal operator associated to Φ as

$$M_\Phi f(x) = \sup_{x\in Q} \|f\|_{\Phi,Q}.$$

An important particular case

$$M_{L(\log L)^\alpha} f(x) = \sup_{x\in Q} \|f\|_{\Phi,Q}$$

where $\Phi(t) = t(1+\log^+ t)^\alpha \qquad \alpha > 0.$

Orlicz maximal functions

Definition

We define the maximal operator associated to Φ as

$$M_\Phi f(x) = \sup_{x\in Q} \|f\|_{\Phi,Q}.$$

An important particular case

$$M_{L(\log L)^\alpha} f(x) = \sup_{x\in Q} \|f\|_{\Phi,Q}$$

where $\Phi(t) = t(1+\log^+ t)^\alpha \qquad \alpha > 0.$

Theorem

For every $k \in \mathbb{N}$ we have that $M_{L(\log L)^k} f(x) \simeq M^{(k+1)} f(x)$

Theorem

Let Φ and Ψ Young functions. If there exists $c > 0$ such that $\Phi(t) \leq \Psi(t) \quad t > c$ then

$$M_\Phi f(x) \lesssim M_\Psi f(x)$$

Since $t \leq \Psi(t) = t(1 + \log^+ t)^\alpha \leq t^r \; r > 1$ for every $\alpha > 0$ we have

$$Mf(x) \lesssim M_\Psi f(x) \leq c_r M_r f(x).$$

Theorem

For every $k \in \mathbb{N}$ *we have that* $M_{L(\log L)^k} f(x) \simeq M^{(k+1)} f(x)$

Theorem

Let Φ *and* Ψ *Young functions. If there exists* $c > 0$ *such that* $\Phi(t) \leq \Psi(t) \qquad t > c$ *then*

$$M_\Phi f(x) \lesssim M_\Psi f(x)$$

Since $t \leq \Psi(t) = t(1 + \log^+ t)^\alpha \leq t^r \ \ r > 1$ for every $\alpha > 0$ we have

$$Mf(x) \lesssim M_\Psi f(x) \leq c_r M_r f(x).$$

Theorem

For every $k \in \mathbb{N}$ *we have that* $M_{L(\log L)^k} f(x) \simeq M^{(k+1)} f(x)$

Theorem

Let Φ *and* Ψ *Young functions. If there exists* $c > 0$ *such that* $\Phi(t) \leq \Psi(t) \quad t > c$ *then*

$$M_\Phi f(x) \lesssim M_\Psi f(x)$$

Since $t \leq \Psi(t) = t(1 + \log^+ t)^\alpha \leq t^r \; r > 1$ for every $\alpha > 0$ we have

$$Mf(x) \lesssim M_\Psi f(x) \leq c_r M_r f(x).$$

Outline

The Conjecture

In [FS] Fefferman and Stein established the following endpoint estimate for arbitrary weights

$$w\left(\{x \in \mathbb{R} : Mf(x) > t\}\right) \lesssim \frac{1}{t} \int_{\mathbb{R}} |f(x)| Mw(x) dx$$

Motivated by that result Muckenhoupt and Wheeden posed the following conjecture

Muckenhoupt-Wheeden conjecture

Let w an arbitrary weight then

$$w\left(\{x \in \mathbb{R} : |Hf(x)| > t\}\right) \lesssim \frac{1}{t} \int_{\mathbb{R}} |f(x)| Mw(x) dx$$

where H stands for the Hilbert transform.

The Conjecture

In [FS] Fefferman and Stein established the following endpoint estimate for arbitrary weights

$$w\left(\{x \in \mathbb{R} : Mf(x) > t\}\right) \lesssim \frac{1}{t} \int_{\mathbb{R}} |f(x)| Mw(x) dx$$

Motivated by that result Muckenhoupt and Wheeden posed the following conjecture

Muckenhoupt-Wheeden conjecture

Let w an arbitrary weight then

$$w\left(\{x \in \mathbb{R} : |Hf(x)| > t\}\right) \lesssim \frac{1}{t} \int_{\mathbb{R}} |f(x)| Mw(x) dx$$

where H stands for the Hilbert transform.

Outline

The development of the conjecture

- First result '70s

$$w\left(\{x\in\mathbb{R}\,:\,|Tf(x)|>t\}\right)\leq\frac{c_r}{t}\int_{\mathbb{R}}|f(x)|M_r w(x)dx$$

- In 1994 Pérez [CP1] established the following result for Calderón-Zygmund operators

$$w\left(\{x\in\mathbb{R}\,:\,|Tf(x)|>t\}\right)\leq\frac{c_\varepsilon}{t}\int_{\mathbb{R}}|f(x)|M_{L(\log L)^\varepsilon}w(x)dx \qquad \varepsilon>0.$$

where $c_\varepsilon\to\infty$ when $\varepsilon\to0$.

- In 2005 Carro, Pérez, J. Soria and F. Soria [CPSS] disproved the conjecture for the fractional integral.
- In 2011 Reguera [R] disproved the conjecture for dyadic case, and in 2012 Reguera and Thiele [RT] disproved the conjecture for the Hilbert transform
- Later on in 2013 A. Criado and F. Soria [CS] disproved the conjecture for the n-dimensional case.

The development of the conjecture

- First result '70s

$$w\left(\{x\in\mathbb{R}\,:\,|Tf(x)|>t\}\right)\leq\frac{c_r}{t}\int_{\mathbb{R}}|f(x)|M_rw(x)dx$$

- In 1994 Pérez [CP1] established the following result for Calderón-Zygmund operators

$$w\left(\{x\in\mathbb{R}\,:\,|Tf(x)|>t\}\right)\leq\frac{c_\varepsilon}{t}\int_{\mathbb{R}}|f(x)|M_{L(\log L)^\varepsilon}w(x)dx\qquad\varepsilon>0.$$

where $c_\varepsilon\to\infty$ when $\varepsilon\to 0$.

- In 2005 Carro, Pérez, J. Soria and F. Soria [CPSS] disproved the conjecture for the fractional integral.
- In 2011 Reguera [R] disproved the conjecture for dyadic case, and in 2012 Reguera and Thiele [RT] disproved the conjecture for the Hilbert transform
- Later on in 2013 A. Criado and F. Soria [CS] disproved the conjecture for the n-dimensional case.

The development of the conjecture

- First result '70s

$$w(\{x \in \mathbb{R} : |Tf(x)| > t\}) \leq \frac{c_r}{t} \int_{\mathbb{R}} |f(x)| M_r w(x) dx$$

- In 1994 Pérez [CP1] established the following result for Calderón-Zygmund operators

$$w(\{x \in \mathbb{R} : |Tf(x)| > t\}) \leq \frac{c_\varepsilon}{t} \int_{\mathbb{R}} |f(x)| M_{L(\log L)^\varepsilon} w(x) dx \qquad \varepsilon > 0.$$

where $c_\varepsilon \to \infty$ when $\varepsilon \to 0$.

- In 2005 Carro, Pérez, J. Soria and F. Soria [CPSS] disproved the conjecture for the fractional integral.
- In 2011 Reguera [R] disproved the conjecture for dyadic case, and in 2012 Reguera and Thiele [RT] disproved the conjecture for the Hilbert transform
- Later on in 2013 A. Criado and F. Soria [CS] disproved the conjecture for the n-dimensional case.

The development of the conjecture

- First result '70s

$$w\left(\{x \in \mathbb{R} : |Tf(x)| > t\}\right) \leq \frac{c_r}{t} \int_{\mathbb{R}} |f(x)| M_r w(x) dx$$

- In 1994 Pérez [CP1] established the following result for Calderón-Zygmund operators

$$w\left(\{x \in \mathbb{R} : |Tf(x)| > t\}\right) \leq \frac{c_\varepsilon}{t} \int_{\mathbb{R}} |f(x)| M_{L(\log L)^\varepsilon} w(x) dx \qquad \varepsilon > 0.$$

  where $c_\varepsilon \to \infty$ when $\varepsilon \to 0$.

- In 2005 Carro, Pérez, J. Soria and F. Soria [CPSS] disproved the conjecture for the fractional integral.

- In 2011 Reguera [R] disproved the conjecture for dyadic case, and in 2012 Reguera and Thiele [RT] disproved the conjecture for the Hilbert transform

- Later on in 2013 A. Criado and F. Soria [CS] disproved the conjecture for the n-dimensional case.

The development of the conjecture

- First result '70s

$$w\left(\{x \in \mathbb{R} : |Tf(x)| > t\}\right) \leq \frac{c_r}{t} \int_{\mathbb{R}} |f(x)| M_r w(x) dx$$

- In 1994 Pérez [CP1] established the following result for Calderón-Zygmund operators

$$w\left(\{x \in \mathbb{R} : |Tf(x)| > t\}\right) \leq \frac{c_\varepsilon}{t} \int_{\mathbb{R}} |f(x)| M_{L(\log L)^\varepsilon} w(x) dx \qquad \varepsilon > 0.$$

  where $c_\varepsilon \to \infty$ when $\varepsilon \to 0$.

- In 2005 Carro, Pérez, J. Soria and F. Soria [CPSS] disproved the conjecture for the fractional integral.
- In 2011 Reguera [R] disproved the conjecture for dyadic case, and in 2012 Reguera and Thiele [RT] disproved the conjecture for the Hilbert transform
- Later on in 2013 A. Criado and F. Soria [CS] disproved the conjecture for the n-dimensional case.

The development of the conjecture

- In 2014 Pérez in a joint work with T. Hytönen [HP]

$$w\left(\{x\in\mathbb{R}\,:\,|Tf(x)|>t\}\right)\leq\frac{c_T}{\varepsilon}\int_{\mathbb{R}}|f(x)|M_{L(\log L)^{\varepsilon}}w(x)dx\quad\varepsilon>0.$$

- In 2015, Domingo-Salazar, Lacey and Rey [DSLR] established the following result

$$w\left(\{x\in\mathbb{R}\,:\,|Tf(x)|>t\}\right)\leq\frac{1}{\varepsilon}\int_{\mathbb{R}}|f(x)|M_{L(\log\log L)^{1+\varepsilon}}w(x)dx\quad\varepsilon>0.$$

- In 2015 Caldarelli, Lerner and Ombrosi [CLO] proved that

$$w\left(\{x\in\mathbb{R}\,:\,|Hf(x)|>t\}\right)\lesssim\frac{1}{t}\int_{\mathbb{R}}|f(x)|M_{\Phi}w(x)dx$$

fails for every $\Phi(t)$ such that $\lim_{t\to\infty}\frac{\Phi(t)}{t\log^+(1+\log^+(t))}=0$

The development of the conjecture

- In 2014 Pérez in a joint work with T. Hytönen [HP]

$$w\left(\{x\in\mathbb{R}\,:\,|Tf(x)|>t\}\right)\leq\frac{c_T}{\varepsilon}\int_{\mathbb{R}}|f(x)|M_{L(\log L)^{\varepsilon}}w(x)dx\quad\varepsilon>0.$$

- In 2015, Domingo-Salazar, Lacey and Rey [DSLR] established the following result

$$w\left(\{x\in\mathbb{R}\,:\,|Tf(x)|>t\}\right)\leq\frac{1}{\varepsilon}\int_{\mathbb{R}}|f(x)|M_{L(\log\log L)^{1+\varepsilon}}w(x)dx\quad\varepsilon>0.$$

- In 2015 Caldarelli, Lerner and Ombrosi [CLO] proved that

$$w\left(\{x\in\mathbb{R}\,:\,|Hf(x)|>t\}\right)\lesssim\frac{1}{t}\int_{\mathbb{R}}|f(x)|M_{\Phi}w(x)dx$$

fails for every $\Phi(t)$ such that $\lim_{t\to\infty}\frac{\Phi(t)}{t\log^{+}(1+\log^{+}(t))}=0$

The development of the conjecture

- In 2014 Pérez in a joint work with T. Hytönen [HP]

$$w\left(\{x \in \mathbb{R} : |Tf(x)| > t\}\right) \leq \frac{c_T}{\varepsilon} \int_{\mathbb{R}} |f(x)| M_{L(\log L)^{\varepsilon}} w(x) dx \quad \varepsilon > 0.$$

- In 2015, Domingo-Salazar, Lacey and Rey [DSLR] established the following result

$$w\left(\{x \in \mathbb{R} : |Tf(x)| > t\}\right) \leq \frac{1}{\varepsilon} \int_{\mathbb{R}} |f(x)| M_{L(\log\log L)^{1+\varepsilon}} w(x) dx \quad \varepsilon > 0.$$

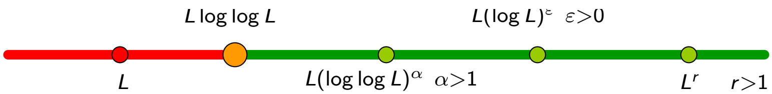
- In 2015 Caldarelli, Lerner and Ombrosi [CLO] proved that

$$w\left(\{x \in \mathbb{R} : |Hf(x)| > t\}\right) \lesssim \frac{1}{t} \int_{\mathbb{R}} |f(x)| M_{\Phi} w(x) dx$$

fails for every $\Phi(t)$ such that $\lim_{t\to\infty} \frac{\Phi(t)}{t\log^{+}(1+\log^{+}(t))} = 0$

Open question

The preceding results can be summarized as follows



Open question

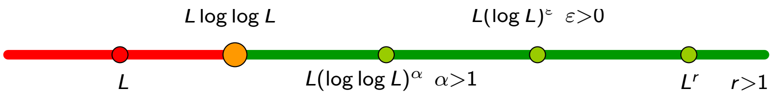
Does the weak type inequality

$$w(\{x \in \mathbb{R} : |Tf(x)| > t\}) \leq c \int_{\mathbb{R}} |f(x)| M_{\Phi} w(x) dx \quad \alpha > 1$$

hold for $\Phi(t) = t \log^{+}(1 + \log^{+}(t))$?

Open question

The preceding results can be summarized as follows



Open question

Does the weak type inequality

$$w\left(\{x \in \mathbb{R} : |Tf(x)| > t\}\right) \leq c \int_{\mathbb{R}} |f(x)| M_\Phi w(x) dx \quad \alpha > 1$$

hold for $\Phi(t) = t\log^+(1+\log^+(t))$?

Outline

The conjecture

For the commutator the natural conjecture is the following

Muckenhoupt-Wheeden conjecture for the commutator

Let T a Calderón-Zygmund operator, $b \in BMO$ and w an arbitrary weight. Then

$$w\left(\{x \in \mathbb{R}^n : |[b,T]f(x)| > t\}\right) \lesssim \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{t}\right) M_{L\log L}w(x)dx.$$

This conjecture is still a conjecture. The methods used in the case of Calderón-Zygmund operators don't seem to apply for this case.

The conjecture

For the commutator the natural conjecture is the following

Muckenhoupt-Wheeden conjecture for the commutator

Let T a Calderón-Zygmund operator, $b \in BMO$ and w an arbitrary weight. Then

$$w\left(\{x \in \mathbb{R}^n : |[b,T]f(x)| > t\}\right) \lesssim \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{t}\right) M_{L\log L}w(x)dx.$$

This conjecture is still a conjecture. The methods used in the case of Calderón-Zygmund operators don't seem to apply for this case.

Outline

Positive results

- In 2001 Pérez and Pradolini [PP] obtained the following estimate

$$w(\{x\in\mathbb{R}^n : |[b,T]f(x)|>t\})\leq c_\varepsilon \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{t}\right) M_{L(\log L)^{1+\varepsilon}} w(x)dx$$

  for every $\varepsilon > 0$, where c_ε is a constant that blows up when $\varepsilon \to 0$.

- In 2011 Ortiz-Caraballo [OC] obtained the following sharp inequality. For every $r > 1$ and every $p > 1$

$$w(\{x\in\mathbb{R}^n : |[b,T]f(x)|>t\})\leq c(pp')^{2p}(r')^{2p-1} \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{t}\right) M_r w(x)dx$$

- Summarizing:

Positive results

- In 2001 Pérez and Pradolini [PP] obtained the following estimate

$$w(\{x\in\mathbb{R}^n : |[b,T]f(x)|>t\})\leq c_\varepsilon \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{t}\right) M_{L(\log L)^{1+\varepsilon}} w(x)dx$$

  for every $\varepsilon > 0$, where c_ε is a constant that blows up when $\varepsilon \to 0$.

- In 2011 Ortiz-Caraballo [OC] obtained the following sharp inequality. For every $r > 1$ and every $p > 1$

$$w(\{x\in\mathbb{R}^n : |[b,T]f(x)|>t\})\leq c(pp')^{2p}(r')^{2p-1} \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{t}\right) M_r w(x)dx$$

- Summarizing:

Positive results

- In 2001 Pérez and Pradolini [PP] obtained the following estimate

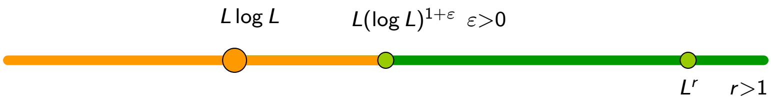
$$w(\{x\in\mathbb{R}^n : |[b,T]f(x)|>t\})\leq c_\varepsilon \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{t}\right) M_{L(\log L)^{1+\varepsilon}} w(x)dx$$

  for every $\varepsilon > 0$, where c_ε is a constant that blows up when $\varepsilon \to 0$.

- In 2011 Ortiz-Caraballo [OC] obtained the following sharp inequality. For every $r > 1$ and every $p > 1$

$$w(\{x\in\mathbb{R}^n : |[b,T]f(x)|>t\})\leq c(pp')^{2p}(r')^{2p-1} \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{t}\right) M_r w(x)dx$$

- Summarizing:



Outline

Our contribution

We have obtained a symbol-multilinear quantitative version of the result by Pérez and Pradolini

Theorem ([PRR])

Let T a Calderón-Zygmund operator, $b \in BMO$ and w an arbitrary weight. Then for every $\varepsilon > 0$

$$w(\{x \in \mathbb{R}^n : |[b,T]f(x)| > t\}) \leq \frac{c}{\varepsilon^2} \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO} \frac{|f(x)|}{t}\right) M_{L(\log L)^{1+\varepsilon}} w(x) dx.$$

Corollary ([OC])

If $w \in A_1$

$$w(\{x \in \mathbb{R}^n : |[b,T]f(x)| > t\}) \leq c\Phi([w]_{A_1})^2 \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO} \frac{|f(x)|}{t}\right) w(x) dx.$$

Outline

Idea of the proof

We follow the scheme of the proof used by Perez & Pradolini and Ortiz-Caraballo, based on Calderón-Zygmund decomposition:

- We control the good part of the function using a sharp strong type inequality.
- We control the bad part of the function using smoothness of the kernel and the following estimate by Hytönen and Pérez [HP]

$$w(\{x \in \mathbb{R} : |Tf(x)| > t\}) \leq \frac{c}{\varepsilon} \int_{\mathbb{R}} |f(x)| M_{L(\log L)^{\varepsilon}} w(x) dx \quad \varepsilon > 0.$$

Idea of the proof

We follow the scheme of the proof used by Perez & Pradolini and Ortiz-Caraballo, based on Calderón-Zygmund decomposition:

- We control the good part of the function using a sharp strong type inequality.
- We control the bad part of the function using smoothness of the kernel and the following estimate by Hytönen and Pérez [HP]

$$w\left(\{x \in \mathbb{R} \, : \, |Tf(x)| > t\}\right) \leq \frac{c}{\varepsilon} \int_{\mathbb{R}} |f(x)| M_{L(\log L)^{\varepsilon}} w(x) dx \quad \varepsilon > 0.$$

Strong type

Theorem

Let T a CZO, $b \in BMO$ and w a weight. There exists a constant c_T depending on T and the dimension such that for every $1 < p < \infty$, $\delta \in (0,1)$

$$\|[b,T]f\|_{L^p(w)} \leq c_T(p'p)^2\left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}\|b\|_{BMO}\|f\|_{L^p(M_\Phi w)}$$

where $\Phi(t) = t(1+\log^+ t)^{2p-1+\delta}$.

Sketch of the proof of the strong type inequality

Let us call $v = M_{L(\log L)^{2p-1+\delta}} w$ and $\kappa = c_T \left(p'p\right)^2 \left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}$. By duality, it suffices to show that

$$\left\| \frac{[b,T]^t f}{v} \right\|_{L^{p'}(v)} \leq \kappa \left\| \frac{f}{w} \right\|_{L^{p'}(w)}$$

Calculating norm by duality we have that

$$\left\| \frac{[b,T]^t f}{v} \right\|_{L^{p'}(v)} = \int_{\mathbb{R}^n} |[b,T]^t f| h dx$$

for some $h \in L^p(v)$

Sketch of the proof of the strong type inequality

Let us call $v = M_{L(\log L)^{2p-1+\delta}} w$ and $\kappa = c_T (p'p)^2 \left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}$. By duality, it suffices to show that

$$\left\| \frac{[b,T]^t f}{v} \right\|_{L^{p'}(v)} \leq \kappa \left\| \frac{f}{w} \right\|_{L^{p'}(w)}$$

Calculating norm by duality we have that

$$\left\| \frac{[b,T]^t f}{v} \right\|_{L^{p'}(v)} = \int_{\mathbb{R}^n} |[b,T]^t f| h dx$$

for some $h \in L^p(v)$

We consider the operator $S(h) = \frac{M\left(hv^{\frac{1}{p}}\right)}{v^{\frac{1}{p}}}$ and build the Rubio de Francia algorithm

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k(h)}{\|S\|_{L^p(v)}^k}$$

R satisfies the following properties:

- $0 \leq h \leq R(h)$
- $\|R(h)\|_{L^p(v)} \leq 2\|h\|_{L^p(v)}$
- $R(h)v^{\frac{1}{p}} \in A_1$ and furthermore $\left[R(h)v^{\frac{1}{p}}\right]_{A_1} \leq cp'$

It's easy to see that $[Rh]_{A_\infty} \leq [Rh]_{A_3} \leq c_n p'$.

Lemma

Let $\delta \in (0,1)$ and $w \in A_\infty$ then

$$\frac{\int_{\mathbb{R}^n} |f(x)| w(x) dx}{\int_{\mathbb{R}^n} M_\delta f(x) w(x) dx} \leq c_{n,\delta}[w]_{A_\infty} \int_{\mathbb{R}^n} M_\delta^\sharp f(x) w(x) dx$$

We consider the operator $S(h) = \frac{M\left(hv^{\frac{1}{p}}\right)}{v^{\frac{1}{p}}}$ and build the Rubio de Francia algorithm

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k(h)}{\|S\|_{L^p(v)}^k}$$

R satisfies the following properties:

- $0 \leq h \leq R(h)$
- $\|R(h)\|_{L^p(v)} \leq 2\|h\|_{L^p(v)}$
- $R(h)v^{\frac{1}{p}} \in A_1$ and furthermore $\left[R(h)v^{\frac{1}{p}}\right]_{A_1} \leq cp'$

It's easy to see that $[Rh]_{A_\infty} \leq [Rh]_{A_3} \leq c_n p'$.

Lemma

Let $\delta \in (0,1)$ and $w \in A_\infty$ then

$$\frac{\int_{\mathbb{R}^n} |f(x)| w(x) dx}{\int_{\mathbb{R}^n} M_\delta f(x) w(x) dx} \leq c_{n,\delta} [w]_{A_\infty} \int_{\mathbb{R}^n} M_\delta^\sharp f(x) w(x) dx$$

We consider the operator $S(h) = \frac{M\left(hv^{\frac{1}{p}}\right)}{v^{\frac{1}{p}}}$ and build the Rubio de Francia algorithm

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k(h)}{\|S\|_{L^p(v)}^k}$$

R satisfies the following properties:

- $0 \leq h \leq R(h)$
- $\|R(h)\|_{L^p(v)} \leq 2\|h\|_{L^p(v)}$
- $R(h)v^{\frac{1}{p}} \in A_1$ and furthermore $\left[R(h)v^{\frac{1}{p}}\right]_{A_1} \leq cp'$

It's easy to see that $[Rh]_{A_\infty} \leq [Rh]_{A_3} \leq c_n p'$.

Lemma

Let $\delta \in (0,1)$ and $w \in A_\infty$ then

$$\frac{\int_{\mathbb{R}^n} |f(x)| w(x) dx}{\int_{\mathbb{R}^n} M_\delta f(x) w(x) dx} \leq c_{n,\delta} [w]_{A_\infty} \int_{\mathbb{R}^n} M_\delta^\sharp f(x) w(x) dx$$

We consider the operator $S(h) = \dfrac{M\left(hv^{\frac{1}{p}}\right)}{v^{\frac{1}{p}}}$ and build the Rubio de Francia algorithm

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k(h)}{\|S\|_{L^p(v)}^k}$$

R satisfies the following properties:

- $0 \leq h \leq R(h)$
- $\|R(h)\|_{L^p(v)} \leq 2\|h\|_{L^p(v)}$
- $R(h)v^{\frac{1}{p}} \in A_1$ and furthermore $\left[R(h)v^{\frac{1}{p}}\right]_{A_1} \leq cp'$

It's easy to see that $[Rh]_{A_\infty} \leq [Rh]_{A_3} \leq c_n p'$.

Lemma

Let $\delta \in (0,1)$ and $w \in A_\infty$ then

$$\frac{\int_{\mathbb{R}^n} |f(x)|w(x)dx}{\int_{\mathbb{R}^n} M_\delta f(x)w(x)dx} \leq c_{n,\delta}[w]_{A_\infty} \int_{\mathbb{R}^n} M_\delta^\sharp f(x)w(x)dx$$

We consider the operator $S(h) = \frac{M\left(hv^{\frac{1}{p}}\right)}{v^{\frac{1}{p}}}$ and build the Rubio de Francia algorithm

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k(h)}{\|S\|_{L^p(v)}^k}$$

R satisfies the following properties:

- $0 \leq h \leq R(h)$
- $\|R(h)\|_{L^p(v)} \leq 2\|h\|_{L^p(v)}$
- $R(h)v^{\frac{1}{p}} \in A_1$ and furthermore $\left[R(h)v^{\frac{1}{p}}\right]_{A_1} \leq cp'$

It's easy to see that $[Rh]_{A_\infty} \leq [Rh]_{A_3} \leq c_n p'$.

Lemma

Let $\delta \in (0,1)$ and $w \in A_\infty$ then

$$\frac{\int_{\mathbb{R}^n} |f(x)|w(x)dx}{\int_{\mathbb{R}^n} M_\delta f(x)w(x)dx} \leq c_{n,\delta}[w]_{A_\infty} \int_{\mathbb{R}^n} M_\delta^\sharp f(x)w(x)dx$$

We consider the operator $S(h) = \frac{M\left(hv^{\frac{1}{p}}\right)}{v^{\frac{1}{p}}}$ and build the Rubio de Francia algorithm

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k(h)}{\|S\|_{L^p(v)}^k}$$

R satisfies the following properties:

- $0 \leq h \leq R(h)$
- $\|R(h)\|_{L^p(v)} \leq 2\|h\|_{L^p(v)}$
- $R(h)v^{\frac{1}{p}} \in A_1$ and furthermore $\left[R(h)v^{\frac{1}{p}}\right]_{A_1} \leq cp'$

It's easy to see that $[Rh]_{A_\infty} \leq [Rh]_{A_3} \leq c_n p'$.

Lemma

Let $\delta \in (0,1)$ and $w \in A_\infty$ then

$$\begin{array}{c}\int_{\mathbb{R}^n} |f(x)|w(x)dx \\ \int_{\mathbb{R}^n} M_\delta f(x)w(x)dx\end{array} \leq c_{n,\delta}[w]_{A_\infty} \int_{\mathbb{R}^n} M_\delta^\sharp f(x)w(x)dx$$

Now we can continue

$$\left\|\frac{[b,T]f}{v}\right\|_{L^{p'}(v)} = \int_{\mathbb{R}^n} |[b,T]^t f| h dx \leq \int_{\mathbb{R}^n} |[b,T]f| Rh dx$$
$$\leq c_n [Rh]_{A_\infty} \int_{\mathbb{R}^n} M_\delta^\sharp([b,T]f) Rh dx$$
$$\leq c_n p' \int_{\mathbb{R}^n} M_\delta^\sharp([b,T]f) Rh dx$$

Lemma(Álvarez, Pérez [AP, CP2])

For $0 < \delta < \varepsilon < \infty$,

$$M_\delta^\sharp([b,T]f)(x) \leq c\|b\|_{BMO} \left(M_\varepsilon(Tf) + M_{L\log L} f\right)$$

and for each $\delta \in (0,1)$,

$$M_\delta^\sharp(Tf)(x) \leq c_\delta M(f)$$

And we have

$$\left\|\frac{[b,T]f}{v}\right\|_{L^{p'}(v)} \leq c_n p' \left(\int_{\mathbb{R}^n} M_{L\log L} f Rh dx + \int_{\mathbb{R}^n} M_\varepsilon(Tf) Rh dx\right) = c_n p' (I_1 + I_2)$$

Now we can continue

$$\left\|\frac{[b,T]f}{v}\right\|_{L^{p'}(v)} = \int_{\mathbb{R}^n} |[b,T]^t f| h dx \leq \int_{\mathbb{R}^n} |[b,T]f| R h dx$$
$$\leq c_n [Rh]_{A_\infty} \int_{\mathbb{R}^n} M_\delta^\sharp([b,T]f) R h dx$$
$$\leq c_n p' \int_{\mathbb{R}^n} M_\delta^\sharp([b,T]f) R h dx$$

Lemma(Álvarez, Pérez [AP, CP2])

For $0 < \delta < \varepsilon < \infty$,

$$M_\delta^\sharp([b,T]f)(x) \leq c\|b\|_{BMO}\left(M_\varepsilon(Tf) + M_{L\log L} f\right)$$

and for each $\delta \in (0,1)$,

$$M_\delta^\sharp(Tf)(x) \leq c_\delta M(f)$$

And we have

$$\left\|\frac{[b,T]f}{v}\right\|_{L^{p'}(v)} \leq c_n p' \left(\int_{\mathbb{R}^n} M_{L\log L} f R h dx + \int_{\mathbb{R}^n} M_\varepsilon(Tf) R h dx\right) = c_n p' (I_1 + I_2)$$

Now we can continue

$$\left\| \frac{[b,T]f}{v} \right\|_{L^{p'}(v)} = \int_{\mathbb{R}^n} |[b,T]^t f| h dx \leq \int_{\mathbb{R}^n} |[b,T]f| R h dx$$
$$\leq c_n [Rh]_{A_\infty} \int_{\mathbb{R}^n} M^\sharp_\delta([b,T]f) R h dx$$
$$\leq c_n p' \int_{\mathbb{R}^n} M^\sharp_\delta([b,T]f) R h dx$$

Lemma(Álvarez, Pérez [AP, CP2])

For $0 < \delta < \varepsilon < \infty$,

$$M^\sharp_\delta([b,T]f)(x) \leq c\|b\|_{BMO} \left(M_\varepsilon(Tf) + M_{L\log L} f \right)$$

and for each $\delta \in (0,1)$,

$$M^\sharp_\delta(Tf)(x) \leq c_\delta M(f)$$

And we have

$$\left\| \frac{[b,T]f}{v} \right\|_{L^{p'}(v)} \leq c_n p' \left(\int_{\mathbb{R}^n} M_{L\log L} f R h dx + \int_{\mathbb{R}^n} M_\varepsilon(Tf) R h dx \right) = c_n p' (I_1 + I_2)$$

Using Hölder inequality and the properties of R,

$$I_1 = \int_{\mathbb{R}^n} M_{L\log L} f(x) Rh(x) dx \leq 2 \left\| \frac{M_{L\log L} f}{v} \right\|_{L^{p'}(v)}$$

Using again both lemmas it's easy to check that

$$I_2 \leq c_n p' \left\| \frac{Mf}{v} \right\|_{L^{p'}(v)}$$

Then

$$\left\| \frac{[b,T]f}{v} \right\|_{L^{p'}(v)} \leq \|b\|_{BMO} c_n (p')^2 \left\| \frac{Mf}{v} \right\|_{L^{p'}(v)}$$

And the proof is reduced to establish the following inequality

$$\left\| \frac{M_{L\log L} f}{v} \right\|_{L^{p'}(v)} \leq c_n p^2 \left(\frac{p-1}{\delta} \right)^{\frac{1}{p'}} \left\| \frac{f}{w} \right\|_{L^{p'}(w)} .$$

Using Hölder inequality and the properties of R,

$$I_1 = \int_{\mathbb{R}^n} M_{L\log L}f(x)Rh(x)dx \leq 2\left\|\frac{M_{L\log L}f}{v}\right\|_{L^{p'}(v)}$$

Using again both lemmas it's easy to check that

$$I_2 \leq c_n p'\left\|\frac{Mf}{v}\right\|_{L^{p'}(v)}$$

Then

$$\left\|\frac{[b,T]f}{v}\right\|_{L^{p'}(v)} \leq \|b\|_{BMO} c_n (p')^2 \left\|\frac{Mf}{v}\right\|_{L^{p'}(v)}$$

And the proof is reduced to establish the following inequality

$$\left\|\frac{M_{L\log L}f}{v}\right\|_{L^{p'}(v)} \leq c_n p^2 \left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}} \left\|\frac{f}{w}\right\|_{L^{p'}(w)}.$$

Using Hölder inequality and the properties of R,

$$I_1 = \int_{\mathbb{R}^n} M_{L\log L} f(x) Rh(x) dx \leq 2 \left\| \frac{M_{L\log L} f}{v} \right\|_{L^{p'}(v)}$$

Using again both lemmas it's easy to check that

$$I_2 \leq c_n p' \left\| \frac{Mf}{v} \right\|_{L^{p'}(v)}$$

Then

$$\left\| \frac{[b,T]f}{v} \right\|_{L^{p'}(v)} \leq \|b\|_{BMO} c_n (p')^2 \left\| \frac{Mf}{v} \right\|_{L^{p'}(v)}$$

And the proof is reduced to establish the following inequality

$$\left\| \frac{M_{L\log L} f}{v} \right\|_{L^{p'}(v)} \leq c_n p^2 \left(\frac{p-1}{\delta} \right)^{\frac{1}{p'}} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}.$$

Using Hölder inequality and the properties of R,

$$I_1 = \int_{\mathbb{R}^n} M_{L\log L} f(x) Rh(x)dx \leq 2 \left\| \frac{M_{L\log L} f}{v} \right\|_{L^{p'}(v)}$$

Using again both lemmas it's easy to check that

$$I_2 \leq c_n p' \left\| \frac{Mf}{v} \right\|_{L^{p'}(v)}$$

Then

$$\left\| \frac{[b,T]f}{v} \right\|_{L^{p'}(v)} \leq \|b\|_{BMO} c_n (p')^2 \left\| \frac{Mf}{v} \right\|_{L^{p'}(v)}$$

And the proof is reduced to establish the following inequality

$$\left\| \frac{M_{L\log L} f}{v} \right\|_{L^{p'}(v)} \leq c_n p^2 \left(\frac{p-1}{\delta} \right)^{\frac{1}{p'}} \left\| \frac{f}{w} \right\|_{L^{p'}(w)} .$$

Since $v = M_{L(\log L)^{2p-1+\delta}} w$, proving

$$\left\| \frac{M_{L(\log L)} f}{v} \right\|_{L^{p'}(v)} \leq c_n p^2 \left(\frac{p-1}{\delta} \right)^{\frac{1}{p'}} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}$$

is equivalent to prove

$$\int M_{L(\log L)} \left(f w^{\frac{1}{p}} \right)^{p'} \left(M_{L(\log L)^{2p-1+\delta}} w \right)^{1-p'} \leq c_n^{p'} p^{2p'} \left(\frac{p-1}{\delta} \right) \int_{\mathbb{R}^n} |f|^{p'}.$$

Idea to prove this inequality

Establish the following inequality

$$M_{L(\log L)} \left(f w^{\frac{1}{p}} \right)(x) \leq c p^2 \left(M_{L(\log L)^{2p-1+\delta}} w(x) \right)^{\frac{1}{p}} M_{\Psi(L)} f(x).$$

Since $v = M_{L(\log L)^{2p-1+\delta}} w$, proving

$$\left\| \frac{M_{L(\log L)} f}{v} \right\|_{L^{p'}(v)} \leq c_n p^2 \left(\frac{p-1}{\delta} \right)^{\frac{1}{p'}} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}$$

is equivalent to prove

$$\int M_{L(\log L)} \left(f w^{\frac{1}{p}} \right)^{p'} \left(M_{L(\log L)^{2p-1+\delta}} w \right)^{1-p'} \leq c_n^{p'} p^{2p'} \left(\frac{p-1}{\delta} \right) \int_{\mathbb{R}^n} |f|^{p'}.$$

Idea to prove this inequality

Establish the following inequality

$$M_{L(\log L)} \left(f w^{\frac{1}{p}} \right) (x) \leq c p^2 \left(M_{L(\log L)^{2p-1+\delta}} w(x) \right)^{\frac{1}{p}} M_{\Psi(L)} f(x).$$

We do that by means of generalized Hölder inequality

Lemma

If Φ_0, Φ_1 and Φ_2 are Young functions such that

$$\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq \kappa\Phi_0^{-1}(x)$$

then

$$\|fg\|_{\Phi_0,Q} \leq 2\kappa\|f\|_{\Phi_1,Q}\|g\|_{\Phi_2,Q}$$

Using a good control of the inverses of the following functions

$$A_\rho(t) = t\left(1+\log^+ t\right)^\rho \qquad X_\rho(t) = \frac{t}{\left(1+\log^+ t\right)^\rho}.$$

The factor $\left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}$ comes from the boundeddness of $M_{\Psi(L)}$ on $L^{p'}$ via B_p condition [HP]

$$\|M_{\Psi(L)}\|_{L^{p'}} \leq c_n\left(\int_1^\infty \frac{\Psi(t)}{t^{p'}}\frac{dt}{t}\right)^{\frac{1}{p'}}.$$

We do that by means of generalized Hölder inequality

Lemma

If Φ_0, Φ_1 and Φ_2 are Young functions such that

$$\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq \kappa\Phi_0^{-1}(x)$$

then

$$\|fg\|_{\Phi_0,Q} \leq 2\kappa\|f\|_{\Phi_1,Q}\|g\|_{\Phi_2,Q}$$

Using a good control of the inverses of the following functions

$$A_\rho(t) = t\left(1+\log^+ t\right)^\rho \qquad X_\rho(t) = \frac{t}{\left(1+\log^+ t\right)^\rho}.$$

The factor $\left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}$ comes from the boundeddness of $M_{\Psi(L)}$ on $L^{p'}$ via B_p condition [HP]

$$\|M_{\Psi(L)}\|_{L^{p'}} \leq c_n\left(\int_1^\infty \frac{\Psi(t)}{t^{p'}}\frac{dt}{t}\right)^{\frac{1}{p'}}.$$

We do that by means of generalized Hölder inequality

Lemma

If Φ_0, Φ_1 and Φ_2 are Young functions such that

$$\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq \kappa\Phi_0^{-1}(x)$$

then

$$\|fg\|_{\Phi_0,Q} \leq 2\kappa\|f\|_{\Phi_1,Q}\|g\|_{\Phi_2,Q}$$

Using a good control of the inverses of the following functions

$$A_\rho(t) = t\left(1 + \log^+ t\right)^\rho \qquad X_\rho(t) = \frac{t}{\left(1 + \log^+ t\right)^\rho}.$$

The factor $\left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}$ comes from the boundeddeness of $M_{\Psi(L)}$ on $L^{p'}$ via B_p condition [HP]

$$\|M_{\Psi(L)}\|_{L^{p'}} \leq c_n\left(\int_1^\infty \frac{\Psi(t)}{t^{p'}}\frac{dt}{t}\right)^{\frac{1}{p'}}.$$

Sketch of the proof of the endpoint estimate

We consider the Calderón-Zygmund decomposition of f . We obtain a family of pairwise disjoint cubes $\{Q_j\}$. If $\Omega = \bigcup_j Q_j$ we can write $f = g + h$ as follows

- $\lambda < \frac{1}{|Q_j|}\int_{Q_j}|f| \leq 2^n\lambda$.
- $g(x) = \begin{cases} f(x) & x \in \Omega^c \\ f_{Q_j} & x \in Q_j \end{cases}, \ |g(x)| \leq 2^n$
- $h = \sum_j h_j$ where $h_j = (f - f_{Q_j})\chi_{Q_j}$.

We denote $\tilde{Q}_j = 5\sqrt{n}Q_j$ and $\tilde{\Omega} = \bigcup_j \tilde{Q}_j$. Then

$$\begin{aligned} w(\{x \in \mathbb{R} : |[b,T]f(x)| > \lambda\}) \leq{} & w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |[b,T]g(x)| > \frac{\lambda}{2}\right\}\right) \\ & + w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |[b,T]h(x)| > \frac{\lambda}{2}\right\}\right) \\ & + w\left(\tilde{\Omega}\right) = I + II + III \end{aligned}$$

Sketch of the proof of the endpoint estimate

We consider the Calderón-Zygmund decomposition of f . We obtain a family of pairwise disjoint cubes $\{Q_j\}$. If $\Omega = \bigcup_j Q_j$ we can write $f = g + h$ as follows

- $\lambda < \frac{1}{|Q_j|}\int_{Q_j}|f| \leq 2^n\lambda.$
- $g(x) = \begin{cases} f(x) & x \in \Omega^c \\ f_{Q_j} & x \in Q_j \end{cases}, \; |g(x)| \leq 2^n$
- $h = \sum_j h_j$ where $h_j = (f - f_{Q_j})\chi_{Q_j}$.

We denote $\tilde{Q}_j = 5\sqrt{n}Q_j$ and $\tilde{\Omega} = \bigcup_j \tilde{Q}_j$. Then

$$\begin{aligned} w\left(\{x \in \mathbb{R} : |[b,T]f(x)| > \lambda\}\right) \leq\; & w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |[b,T]g(x)| > \frac{\lambda}{2}\right\}\right) \\ & + w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |[b,T]h(x)| > \frac{\lambda}{2}\right\}\right) \\ & + w\left(\tilde{\Omega}\right) = I + II + III \end{aligned}$$

Sketch of the proof of the endpoint estimate

We consider the Calderón-Zygmund decomposition of f . We obtain a family of pairwise disjoint cubes $\{Q_j\}$. If $\Omega = \bigcup_j Q_j$ we can write $f = g + h$ as follows

- $\lambda < \frac{1}{|Q_j|}\int_{Q_j}|f| \leq 2^n\lambda$.
- $g(x) = \begin{cases} f(x) & x \in \Omega^c \\ f_{Q_j} & x \in Q_j \end{cases}$, $|g(x)| \leq 2^n$
- $h = \sum_j h_j$ where $h_j = (f - f_{Q_j})\chi_{Q_j}$.

We denote $\tilde{Q}_j = 5\sqrt{n}Q_j$ and $\tilde{\Omega} = \bigcup_j \tilde{Q}_j$. Then

$$w(\{x \in \mathbb{R} : |[b,T]f(x)| > \lambda\}) \leq w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |[b,T]g(x)| > \frac{\lambda}{2}\right\}\right) + w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |[b,T]h(x)| > \frac{\lambda}{2}\right\}\right) + w\left(\tilde{\Omega}\right) = I + II + III$$

Sketch of the proof of the endpoint estimate

We consider the Calderón-Zygmund decomposition of f . We obtain a family of pairwise disjoint cubes $\{Q_j\}$. If $\Omega = \bigcup_j Q_j$ we can write $f = g + h$ as follows

- $\lambda < \frac{1}{|Q_j|}\int_{Q_j} |f| \leq 2^n\lambda$.
- $g(x) = \begin{cases} f(x) & x \in \Omega^c \\ f_{Q_j} & x \in Q_j \end{cases}, \ |g(x)| \leq 2^n$
- $h = \sum_j h_j$ where $h_j = (f - f_{Q_j})\chi_{Q_j}$.

We denote $\tilde{Q}_j = 5\sqrt{n}Q_j$ and $\tilde{\Omega} = \bigcup_j \tilde{Q}_j$. Then

$$w(\{x \in \mathbb{R} : |[b,T]f(x)| > \lambda\}) \leq w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |[b,T]g(x)| > \frac{\lambda}{2}\right\}\right) + w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |[b,T]h(x)| > \frac{\lambda}{2}\right\}\right) + w\left(\tilde{\Omega}\right) = I + II + III$$

Sketch of the proof of the endpoint estimate

We consider the Calderón-Zygmund decomposition of f . We obtain a family of pairwise disjoint cubes $\{Q_j\}$. If $\Omega = \bigcup_j Q_j$ we can write $f = g + h$ as follows

- $\lambda < \frac{1}{|Q_j|}\int_{Q_j} |f| \leq 2^n\lambda$.
- $g(x) = \begin{cases} f(x) & x \in \Omega^c \\ f_{Q_j} & x \in Q_j \end{cases}, \ |g(x)| \leq 2^n$
- $h = \sum_j h_j$ where $h_j = (f - f_{Q_j})\chi_{Q_j}$.

We denote $\tilde{Q}_j = 5\sqrt{n}Q_j$ and $\tilde{\Omega} = \bigcup_j \tilde{Q}_j$. Then

$$
\begin{aligned}
w(\{x \in \mathbb{R} : |[b,T]f(x)| > \lambda\}) &\leq w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |[b,T]g(x)| > \frac{\lambda}{2}\right\}\right) \\
&+ w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |[b,T]h(x)| > \frac{\lambda}{2}\right\}\right) \\
&+ w\left(\tilde{\Omega}\right) = I + II + III
\end{aligned}
$$

Control of the good part

By a standard proccedure it's easy to see that

$$III \leq c_n \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} Mw(y)dy$$

To control I we use the strong type. We take

$$1+\frac{\varepsilon}{6} < p < 1+\frac{\varepsilon}{4} \qquad \delta = \varepsilon - 2(p-1).$$

For that choice of p and δ we have that

$$(p')^{2p}p^{2p}\left(\frac{p-1}{\delta}\right)^{\frac{p}{p'}} \leq c\frac{1}{\varepsilon^2} \qquad 2p-1+\delta = 1+\varepsilon.$$

Using the strong type with that choices for p and δ

$$I \leq c\frac{1}{\varepsilon^2}\int_{\mathbb{R}^n}\frac{|f(y)|}{\lambda}M_{L(\log L)^{1+\varepsilon}}w(y)dy.$$

Control of the good part

By a standard proccedure it's easy to see that

$$III \leq c_n \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} Mw(y)dy$$

To control I we use the strong type. We take

$$1+\frac{\varepsilon}{6} < p < 1+\frac{\varepsilon}{4} \qquad \delta = \varepsilon - 2(p-1).$$

For that choice of p and δ we have that

$$(p')^{2p}p^{2p}\left(\frac{p-1}{\delta}\right)^{\frac{p}{p'}} \leq c\frac{1}{\varepsilon^2} \qquad 2p-1+\delta = 1+\varepsilon.$$

Using the strong type with that choices for p and δ

$$I \leq c\frac{1}{\varepsilon^2}\int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} M_{L(\log L)^{1+\varepsilon}} w(y)dy.$$

Control of the good part

By a standard proccedure it's easy to see that

$$III \leq c_n \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} Mw(y)dy$$

To control I we use the strong type. We take

$$1+\frac{\varepsilon}{6} < p < 1+\frac{\varepsilon}{4} \qquad \delta = \varepsilon - 2(p-1).$$

For that choice of p and δ we have that

$$(p')^{2p} p^{2p} \left(\frac{p-1}{\delta}\right)^{\frac{p}{p'}} \leq c\frac{1}{\varepsilon^2} \qquad 2p-1+\delta = 1+\varepsilon.$$

Using the strong type with that choices for p and δ

$$I \leq c\frac{1}{\varepsilon^2} \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} M_{L(\log L)^{1+\varepsilon}} w(y)dy.$$

Control of the bad part

To control II, firstly we write

$$[b,T]h = \sum_j (b - b_{Q_j}) T h_j - \sum_j T\left((b - b_{Q_j}) h_j\right)$$

then

$$\begin{aligned} II &\leq w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : \left|\sum_j (b(x) - b_{Q_j}) T h_j(x)\right| > \frac{\lambda}{4}\right\}\right) \\ &+ w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : \left|\sum_j T\left((b - b_{Q_j}) h_j\right)(x)\right| > \frac{\lambda}{4}\right\}\right) \\ &= A + B \end{aligned}$$

The control of A relies on the smoothness of the kernel, and the generalized Hölder inequality. We obtain that

$$A \leq c \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} M_{L \log L} w(y) dy$$

Control of the bad part

To control II, firstly we write

$$[b,T]h=\sum_j (b-b_{Q_j})Th_j-\sum_j T\left((b-b_{Q_j})h_j\right)$$

then

$$\begin{aligned}II &\leq w\left(\left\{x\in\mathbb{R}^n\setminus\tilde{\Omega}\,:\,\left|\sum_j (b(x)-b_{Q_j})Th_j(x)\right|>\frac{\lambda}{4}\right\}\right)\\ &+w\left(\left\{x\in\mathbb{R}^n\setminus\tilde{\Omega}\,:\,\left|\sum_j T\left((b-b_{Q_j})h_j\right)(x)\right|>\frac{\lambda}{4}\right\}\right)\\ &=A+B\end{aligned}$$

The control of A relies on the smoothness of the kernel, and the generalized Hölder inequality. We obtain that

$$A\leq c\int_{\mathbb{R}^n}\frac{|f(y)|}{\lambda}M_{L\log L}w(y)dy$$

To control II, firstly we write

$$[b, T]h = \sum_j (b - b_{Q_j}) T h_j - \sum_j T\left((b - b_{Q_j}) h_j\right)$$

then

$$\begin{aligned} II &\leq w\left(\left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega} \, : \, \left| \sum_j (b(x) - b_{Q_j}) T h_j(x) \right| > \frac{\lambda}{4} \right\}\right) \\ &+ w\left(\left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega} \, : \, \left| \sum_j T\left((b - b_{Q_j}) h_j\right)(x) \right| > \frac{\lambda}{4} \right\}\right) \\ &= A + B \end{aligned}$$

The control of A relies on the smoothness of the kernel, and the generalized Hölder inequality. We obtain that

$$A \leq c \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} M_{L \log L} w(y) dy$$

Control of the bad part

To control B, using Hytönen-Pérez result

$$w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : \left|\sum_j T\left((b - b_{Q_j})h_j\right)(x)\right| > \frac{\lambda}{4}\right\}\right)$$

$$\leq \frac{1}{\varepsilon}\frac{1}{\lambda}\int_{\mathbb{R}^n}\left|\sum_j (b(x) - b_{Q_j})h_j(x)\right| M_{L(\log L)^\varepsilon}\left(w\chi_{\mathbb{R}^n\setminus\tilde{\Omega}}\right)(x)dx$$

Using the properties of the Calderón-Zygmund cubes and the alternative definition of the Orlicz norm

$$\|f\|_{\Phi,Q} \simeq \inf_{\mu>0}\left\{\mu + \frac{\mu}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\mu}\right)dx\right\}$$

we bound the latter by

$$\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) M_{L(\log L)^\varepsilon} w(x)dx.$$

To control B, using Hytönen-Pérez result

$$w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : \left|\sum_j T\left((b - b_{Q_j})h_j\right)(x)\right| > \frac{\lambda}{4}\right\}\right)$$

$$\leq \frac{1}{\varepsilon}\frac{1}{\lambda}\int_{\mathbb{R}^n}\left|\sum_j (b(x) - b_{Q_j})h_j(x)\right| M_{L(\log L)^\varepsilon}\left(w\chi_{\mathbb{R}^n\setminus\tilde{\Omega}}\right)(x)dx$$

Using the properties of the Calderón-Zygmund cubes and the alternative definition of the Orlicz norm

$$\|f\|_{\Phi,Q} \simeq \inf_{\mu>0}\left\{\mu + \frac{\mu}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\mu}\right)dx\right\}$$

we bound the latter by

$$\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\lambda}\right)M_{L(\log L)^\varepsilon}w(x)dx.$$

Work in progress

In a work in progress with Sheldy Ombrosi and Andrei Lerner, it seems we can obtain the following estimates

$$\begin{aligned} w\left(\{x \in \mathbb{R}^n : |[b,T]f(x)| > t\}\right) &\lesssim \frac{c}{\varepsilon}\int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{t}\right) M_{L(\log L)^{1+\varepsilon}}w(x)dx \\ &\lesssim \frac{c}{\varepsilon}\int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{t}\right) M_{L(\log L)(\log\log L)^{1+\varepsilon}}w(x)dx \end{aligned}$$

Our proof relies on a suitable pointwise control for the commutator and in an adaptation of the arguments given in [DSLR].
This also leads to an improvement on the dependence on the A_1 constant, namely

$$w\left(\{x \in \mathbb{R}^n : |[b,T]f(x)| > t\}\right) \leq [w]_{A_1}^2 \log(e+[w]_{A_1}) \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO}\frac{|f(x)|}{t}\right) w(x)dx$$

References I

[AP] Alvarez, J.; Pérez, C. Estimates with A_∞ weights for various singular integral operators. Boll. Un. Mat. Ital. A (7) 8 (1994), no. 1, 123–133.

[CPSS] Carro, M.J.; Pérez, C., Soria, F., Soria, J.,Maximal functions and the control of weighted inequalities for the fractional integral operator. Indiana University Mathematics Journal, 54 (2005), 627–644.

[CLO] Marcela Caldarelli, Andrei K. Lerner, Sheldy Ombrosi On a counterexample related to weighted weak type estimates for singular integrals arXiv:1506.08317

[CRW] Coifman, R. R.; Rochberg, R.; Weiss, Guido Factorization theorems for Hardy spaces in several variables. Ann. of Math. (2) 103 (1976), no. 3, 611–635.

[CS] Alberto Criado, Fernando Soria. Muckenhoupt-Wheeden conjectures in higher dimensions. arXiv:1312.5255

[DSLR] Carlos Domingo-Salazar, Michael T. Lacey, Guillermo Rey Borderline Weak Type Estimates for Singular Integrals and Square Functions arXiv:1505.01804

References II

[FS] C. Fefferman, E.M. Stein, Some maximal inequalities, Amer. J. Math. 93 (1971) 107–115.

[HP] Hytönen, Tuomas; Pérez, Carlos The $L(logL)^{\varepsilon}$ endpoint estimate for maximal singular integral operators. J. Math. Anal. Appl. 428 (2015), no. 1, 605–626.

[OC] Ortiz-Caraballo, Carmen Quadratic A_1 bounds for commutators of singular integrals with BMO functions. Indiana Univ. Math. J. 60 (2011), no. 6, 2107–2129.

[CP1] Pérez, C. Weighted norm inequalities for singular integral operators. J. London Math. Soc. (2) 49 (1994), no. 2, 296–308.

[CP2] Pérez, Carlos Endpoint estimates for commutators of singular integral operators. J. Funct. Anal. 128 (1995), no. 1, 163–185.

[PP] Pérez, Carlos; Pradolini, Gladis Sharp weighted endpoint estimates for commutators of singular integrals. Michigan Math. J. 49 (2001), no. 1, 23–37.

[PRR] Carlos Pérez, Israel P. Rivera-Ríos Borderline weighted estimates for commutators of singular integrals. arXiv:1507.08568

References III

[R] Reguera, Maria Carmen On Muckenhoupt-Wheeden conjecture. Adv. Math. 227 (2011), no. 4, 1436–1450.

[RT] Reguera, Maria Carmen; Thiele, Christoph The Hilbert transform does not map $L^1(Mw)$ to $L^{1,\infty}(w)$. Math. Res. Lett. 19 (2012), no. 1, 1–7.

THANK
YOU
GRACIAS
ARIGATO
SHUKURIA
DANKSCHEEN
GRÀCIES
TASHAKKUR ATU
YAQHANYELAY
KIITOS
TINGKI
BİYAN
SHUKRIA
SUKSAMA
EKHMET
MEHRBANI
GRAZIE
MAAKE
KOMAPSUMNIDA
PALDIES
BOLZÏN
MERCI
GRAZAS
GOZAIMASHITA
EFCHARISTO
ESKERRIK ASKO
JUSPAXAR
SPASSIBO
SNACHALHUYA
NUHUN
CHALTU
WABEEJA
MAITEKA
SPASIBO
DENKAUJA
NENACHALHYA
MERSI
ATTO
LAH
SIKOMO
MAKETAI
MINMONCHAR
TAVTAPUCH
MEDAWAGSE
MERASTAWHY
GAEJTHO
AGUYJE
FAKAAUE